A SURVEY OF $J$-SPACES

E. MICHAEL


1. Basic concepts

A space $X$ is a $J$-space if, whenever $\{A, B\}$ is a closed cover of $X$ with $A \cap B$ compact, then $A$ or $B$ is compact. A space $X$ is a strong $J$-space if every compact $K \subset X$ is contained in a compact $L \subset X$ with $X \setminus L$ connected. [As in [4], all maps are continuous and all spaces are Hausdorff.]

1.1. Every strong $J$-space $X$ is a $J$-space. The two concepts coincide when $X$ is locally connected, but in general (even for closed subsets of $\mathbb{R}^2$) they do not.

2. Examples

2.1. A topological linear space $X$ is a (strong) $J$-space if and only if $X \neq \mathbb{R}$.

2.2. If $X$ and $Y$ are connected and non-compact, then $X \times Y$ is a strong $J$-space.\textsuperscript{1}

2.3. Let $Y$ be a compact manifold with boundary $B$, and let $A \subset B$. Then $Y \setminus A$ is a (strong) $J$-space if and only if $A$ is connected.

\textsuperscript{1}This was proved in [5].
3. Characterizations by closed maps

A map \( f : X \to Y \) is called boundary-perfect if \( f \) is closed and \( \text{bdry} \, f^{-1}(y) \) is compact for every \( y \in Y \). It follows from [3] that every closed map \( f : X \to Y \) from a paracompact space \( X \) to a \( q \)-space \( Y \) is boundary-perfect.\(^2\)

3.1. A space \( X \) is a \( J \)-space if and only if every boundary-perfect map \( f : X \to Y \) onto a non-compact space \( Y \) is perfect.

3.2. If \( X \) is a \( J \)-space, then every boundary-perfect map \( f : X \to Y \) has at most one non-compact fiber. The converse holds if \( X \) is locally compact.

3.3. Let \( X \) be paracompact and locally compact. Then the following are equivalent.

(a) \( X \) is a \( J \)-space.
(b) Every closed map \( f : X \to Y \) onto a non-compact, locally compact space \( Y \) is perfect.
(c) Every closed map \( f : X \to Y \) onto a locally compact space \( Y \) has at most one non-compact fiber.

3.4. Let \( X \) be metrizable. Then the following are equivalent

(a) \( X \) is a \( J \)-space.
(b) Every closed map \( f : X \to Y \) onto a non-compact, metrizable space \( Y \) is perfect.

4. Characterization by compactifications

Call a set \( A \subset Y \) a boundary set for \( Y \) if \( \text{Int} \, A = \emptyset \) and, whenever \( U \supset A \) is open in \( Y \) and \( \{W_1, W_2\} \) is a disjoint, relatively open cover of \( U \setminus A \), then no \( y \in A \) lies in \( \overline{W_1 \cap W_2} \). Call a set \( A \subset Y \) a strong boundary set for \( Y \) if \( \text{Int} \, A = \emptyset \) and, whenever \( U \supset A \) is open in \( Y \), then every \( y \in A \) has an open neighborhood \( V \subset U \) with \( V \setminus A \) connected.

It is easy to see that, if \( Y \) is a manifold with boundary \( B \), then every \( A \subset B \) is a strong boundary set for \( Y \). And it follows from the proof of [1, Lemma 4] (or from [2, Proposition 3.5]) that, if \( Y \) is completely regular, then \( \beta X \setminus X \) is a boundary set for \( \beta X \).

4.1. Let \( Y \) be a compactification of \( X \), and suppose either that \( X \) is locally compact or that \( Y \) is metrizable. Then the following are equivalent.

(a) \( X \) is a (strong) \( J \)-space.
(b) \( Y \setminus X \) is connected and a (strong) boundary set for \( Y \).

5. Preservation

5.1. \( J \)-spaces are preserved by boundary-perfect images. (False for strong \( J \)-spaces, even with perfect images.)

\(^{q}\)\(q\)-spaces (see [3]) include all locally compact and all metrizable spaces.
5.2. $J$-spaces and strong $J$-spaces are preserved by monotone, perfect pre-images.

5.3. If $X_1$, $X_2$ are connected, then $X_1 \times X_2$ is a (strong) $J$-space if and only if either $X_1$, $X_2$ are both (strong) $J$-spaces or both are non-compact.

5.4. Let $\{X_1, X_2\}$ be a closed cover of $X$ with $X_1 \cap X_2$ compact. Then $X$ is a (strong) $J$-space if and only if $X_1$, $X_2$ are both (strong) $J$-spaces and $X_1$ or $X_2$ is compact.

5.5. If $X$ is a (strong) $J$-space, so is every component of $X$. (False for $J$-spaces).

References


University of Washington, Seattle, WA, U.S.A.