FELL-CONTINUOUS SELECTIONS AND TOPOLOGICALLY WELL-ORDERABLE SPACES II

VALENTIN GUTEV

Abstract. The present paper improves a result of [3] by showing that a space $X$ is topologically well-orderable if and only if there exists a selection for $F_2(X)$ which is continuous with respect to the Fell topology on $F_2(X)$. In particular, this implies that $F(X)$ has a Fell-continuous selection if and only if $F_2(X)$ has a Fell-continuous selection.

1. Introduction

Let $X$ be a topological space, and let $F(X)$ be the family of all non-empty closed subsets of $X$. Also, let $\tau$ be a topology on $F(X)$ and $D \subset F(X)$. A map $f : D \to X$ is a selection for $D$ if $f(S) \in S$ for every $S \in D$. A map $f : D \to X$ is a $\tau$-continuous selection for $D$ if it is a selection for $D$ which is continuous with respect to the relative topology $\tau$ on $D$ as a subspace of $F(X)$.

Two topologies on $F(X)$ will play the most important role in this paper. The first one is the Vietoris topology $\tau_V$ which is generated by all collections of the form

$$\langle V \rangle = \left\{ S \in F(X) : S \cap V \neq \emptyset, V \in \mathcal{V}, \text{ and } S \subset \bigcup \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of $X$. The other one is the Fell topology $\tau_F$ which is defined by all basic Vietoris neighbourhood $\langle V \rangle$ with the property that $X \setminus \bigcup \mathcal{V}$ is compact.

Finally, let us recall that a space $X$ is topologically well-orderable (see Engelking, Heath and Michael [2]) if there exists a linear order “$\prec$” on $X$ such that $X$ is a linear ordered topological space with respect to $\prec$, and every non-empty closed subset of $X$ has a $\prec$-minimal element.

2000 Mathematics Subject Classification. Primary 54B20, 54C65; Secondary 54D45, 54F05.

Key words and phrases. Hyperspace topology, selection, ordered space, local compactness.
Recently, the topologically well-orderable spaces were characterized in [3, Theorem 1.3] by means of Fell-continuous selections for their hyperspaces of non-empty closed subsets.

**Theorem 1.1** ([3]). *A Hausdorff space $X$ is topologically well-orderable if and only if $\mathcal{F}(X)$ has a $\tau_F$-continuous selection.*

In the present paper, we improve Theorem 1.1 by showing that one may use $\tau_F$-continuous selections only for the subset $\mathcal{F}_2(X) = \{S \in \mathcal{F}(X) : |S| \leq 2\}$ of $\mathcal{F}(X)$. Namely, the following theorem will be proven.

**Theorem 1.2.** *A Hausdorff space $X$ is topologically well-orderable if and only if $\mathcal{F}_2(X)$ has a $\tau_F$-continuous selection.*

About related results for Vietoris-continuous selections, the interested reader is referred to van Mill and Wattel [6].

Theorem 1.2 is interesting also from another point of view. According to Theorem 1.1, it implies the following result which may have an independent interest.

**Corollary 1.3.** *If $X$ is a Hausdorff space, then $\mathcal{F}(X)$ has a $\tau_F$-continuous selection if and only if $\mathcal{F}_2(X)$ has a $\tau_F$-continuous selection.*

A word should be said also about the proof of Theorem 1.2. In general, it is based on the proof of Theorem 1.1 stated in [3], and is separated in a few different steps which are natural generalizations of the corresponding ones given in [3]. In fact, the paper demonstrates that all statements of [3] remain true if $\mathcal{F}(X)$ is replaced by $\mathcal{F}_2(X)$. Related to this, the interested reader may consult an alternative proof of Theorem 1.2 given in [1] and based again on the scheme in [3].

2. A REDUCTION TO LOCALLY COMPACT SPACES

In the sequel, all spaces are assumed to be at least Hausdorff.

In this section, we prove the following generalization of [3, Theorem 2.1].

**Theorem 2.1.** *Let $X$ be a space such that $\mathcal{F}_2(X)$ has a $\tau_F$-continuous selection. Then $X$ is locally compact.*

**Proof.** We follow the proof of [3, Theorem 2.1]. Namely, let $f$ be a $\tau_F$-continuous selection for $\mathcal{F}_2(X)$ and suppose, if possible, that $X$ is not locally compact. Hence, there exists a point $p \in X$ such that $\mathcal{V}$ is not compact for every neighbourhood $V$ of $p$ in $X$. Claim that there exists a point $q \in X$ such that

(1) \hspace{1cm} $q \neq p$ and $f(\{p, q\}) = p$.

To this purpose, note that there exists $F \in \mathcal{F}(X)$ such that $F$ is not compact and $p \notin F$. Then, $f^{-1}(X \setminus F)$ is a $\tau_F$-neighbourhood of $\{p\}$ in $\mathcal{F}_2(X)$, so
there exists a finite family $W$ of open subsets of $X$ such that $X \setminus \bigcup W$ is compact and

$$\{p\} \in (W) \cap \mathcal{F}_2(X) \subset f^{-1}(X \setminus F).$$

Then, $F \cap W \neq \emptyset$ for some $W \in W$ because $F$ is not compact. Therefore, there exists a point $q \in F \cap (\bigcup W)$. This $q$ is as required.

Let $q$ be as in (1). Since $X$ is Hausdorff, $f(\{q\}) \neq f(\{p, q\})$, and $f$ is $\tau_F$-continuous, there now exist two finite families $\mathcal{U}$ and $\mathcal{V}$ of open subsets of $X$ such that $X \setminus \bigcup \mathcal{U}$ is compact, $\{q\} \in (\mathcal{U})$, $\{p, q\} \in (\mathcal{V})$, and $(\mathcal{U}) \cap (\mathcal{V}) = \emptyset$. Then,

$$\text{(2)} \quad p \in V_p = \bigcap\{V \in \mathcal{V} : p \in V\} \subset X \setminus \bigcup \mathcal{U}.$$  

Indeed, suppose there is a point $\ell \in V_p \cap (\bigcup \mathcal{U})$. Then, $\{\ell, q\} \in (\mathcal{U})$ because $\{q\} \in (\mathcal{U})$. However, we also get that $\{\ell, q\} \in (\mathcal{V})$ because $q \notin V$ for some $V \in \mathcal{V}$ implies $p \in V$, hence $\ell \in V_p \subset V$. Thus, we finally get that $\{\ell, q\} \in (\mathcal{U}) \cap (\mathcal{V})$ which is impossible. So, (2) holds as well.

To finish the proof, it remains to observe that this contradicts the choice of $p$. Namely $V_p$ becomes a neighbourhood of $p$ which, by (2), has a compact closure because $X \setminus \bigcup \mathcal{U}$ is compact.

\[\square\]

3. A reduction to compact spaces

For a locally compact space $X$ we will use $\alpha X$ to denote the one point compactification of $X$. For a non-compact locally compact $X$ let us agree to denote by $\alpha$ the point of the singleton $\alpha X \setminus X$.

In what follows, to every family $D \subset \mathcal{F}(X)$ we associate a family $\alpha(D) \subset \mathcal{F}(\alpha X)$ defined by

$$\alpha(D) = \{S \in \mathcal{F}(\alpha X) : S \cap X \in D \cup \{\emptyset\}\}.$$  

The following extension theorem was actually proven in [3, Theorem 3.1].

**Theorem 3.1.** Let $X$ be a locally compact non-compact space $X$, and $D \subset \mathcal{F}(X)$. Then, $D$ has a $\tau_F$-continuous selection if and only if $\alpha(D)$ has a $\tau_V$-continuous selection $g$ such that $g^{-1}(\alpha) = \{\alpha\}$.

**Proof.** Just the same proof as in [3, Theorem 3.1] works. Namely, if $f$ is a $\tau_F$-continuous selection for $D$, we may define a selection $g$ for $\alpha(D)$ by $g(S) = f(S \cap X)$ if $S \cap X \neq \emptyset$ and $g(S) = \alpha$ otherwise, where $S \in \alpha(D)$. Clearly $g^{-1}(\alpha) = \{\alpha\}$ and, as shown in [3, Theorem 3.1], $g$ is $\tau_V$-continuous. If now $g$ is a $\tau_V$-continuous selection for $\alpha(D)$, with $g^{-1}(\alpha) = \{\alpha\}$, then $g(S \cup \{\alpha\}) \in S$ for every $S \in D$, so we may define a selection $f$ for $D$ by $f(S) = g(S \cup \{\alpha\})$, $S \in D$. The verification that $f$ is $\tau_F$-continuous was done in [3, Theorem 3.1].  

\[\square\]
4. Special selections and connected sets

In what follows, to every selection \( f : \mathcal{F}_2(X) \to X \) we associate an order-like relation "\( \prec_f \)" on \( X \) (see Michael [5]) defined for \( x \neq y \) by

\[
x_1 \prec_f x_2 \iff f(\{x_1, x_2\}) = x_1.
\]

Further, we will need also the following \( \prec_f \)-intervals:

\[
(x, +\infty)_{\prec_f} = \{z \in X : x \prec_f z\}
\]

and

\[
[x, +\infty)_{\prec_f} = \{z \in X : x \leq_f z\}.
\]

Now, we provide the generalization of [3, Theorem 4.1] for the case of \( \mathcal{F}_2(X) \).

**Theorem 4.1.** Let \( X \) be a space, \( a \in X \), and let \( A \in \mathcal{F}(X) \) be a connected set such that \( |A| > 1 \) and \( a \in A \setminus \overline{A} \). Also, let \( f : \mathcal{F}_2(X) \to X \) be a \( \tau_V \)-continuous selection for \( \mathcal{F}_2(X) \). Then, \( f^{-1}(a) \neq \{a\} \).

**Proof.** Suppose, on the contrary, that \( f^{-1}(a) = \{a\} \). By hypothesis, there exists a point \( b \in A \), with \( b \neq a \). Since \( f \) is \( \tau_V \)-continuous, \( f(\{a, b\}) = b \) and \( a \in \overline{X \setminus A} \), we can find a point \( c \in X \setminus A \) such that \( f(\{b, c\}) = b \). Then, \( B = A \cap (c, +\infty)_{\prec_f} \) is a clopen subset of \( A \) because \( B = A \cap [c, +\infty)_{\prec_f} \), see [5]. However, this is impossible because \( b \in A \setminus B \), while \( a \in B \). \( \Box \)

5. A further result about special selections

Following [3], we shall say that a point \( a \in X \) is a partition of \( X \) if there are open subset \( L, R \subset X \setminus \{a\} \) such that \( L \cap R = \emptyset \) and \( L \cap R = \emptyset \).

We finalize the preparation for the proof of Theorem 1.2 with the following result about special Vietoris continuous selections and partitions which generalizes [3, Theorem 5.1].

**Theorem 5.1.** Let \( X \) be a compact space, \( f \) a \( \tau_V \)-continuous selection for \( \mathcal{F}_2(X) \), and let \( a \in X \) be a partition of \( X \) such that \( f^{-1}(a) = \{a\} \). Then, \( X \) is first countable at \( a \).

**Proof.** By definition, there are open sets \( L, R \subset X \setminus \{a\} \) such that \( \overline{L} \cap \overline{R} = \{a\} \) and \( L \cap R = \emptyset \). Hence, both \( L \) and \( R \) are non-empty. Take a point \( \ell_0 \in L \). Then, by hypothesis, \( f(\{\ell_0, a\}) = \ell_0 \). Since \( f \) is \( \tau_V \)-continuous, this implies the existence of a neighbourhood \( L_0 \subset L \) of \( \ell_0 \) and a neighbourhood \( V_0 \) of \( a \) such that

\[
L_0 \cap V_0 = \emptyset \text{ and } f(\{L_0, V_0\}) \subset \mathcal{F}_2(X) \subset L_0.
\]

Since \( a \in \overline{R} \), there exists a point \( r_0 \in V_0 \cap R \). Observe that \( f(\{a, r_0\}) = r_0 \in V_0 \). Hence, just like before, we may find a neighbourhood \( R_0 \subset R \cap V_0 \) of \( r_0 \) and a neighbourhood \( W_0 \subset V_0 \) of \( a \) such that

\[
R_0 \cap W_0 = \emptyset \text{ and } f(\{R_0, W_0\}) \subset \mathcal{F}_2(X) \subset R_0.
\]
Thus, by induction, we may construct a sequence \( \{ \ell_n : n < \omega \} \) of points of \( L \), a sequence \( \{ r_n : n < \omega \} \) of points of \( R \), and open sets \( L_n, V_n, R_n, W_n \subset X \) such that

\[
\begin{align*}
\ell_n & \in L_n, \\
a & \in V_n, \\
L_n \cap V_n & = \emptyset \text{ and } f(\{L_n, V_n\} \cap F_2(X)) \subset L_n, \\
r_n & \in R_n, \\
a & \in W_n, \\
R_n \cap W_n & = \emptyset \text{ and } f(\{R_n, W_n\} \cap F_2(X)) \subset R_n,
\end{align*}
\]

and

\[
\begin{align*}
V_{n+1} & \subset W_n \subset V_n, \\
L_{n+1} & \subset L \cap W_n \text{ and } R_n \subset R \cap V_n.
\end{align*}
\]

Since \( X \) is compact, \( \{ \ell_n : n < \omega \} \) has a cluster point \( \ell \), and \( \{ r_n : n < \omega \} \) has a cluster point \( r \). We claim that \( \ell = r \). Indeed, suppose for instance that \( \ell \not< f r \) (the case \( r \not< f \ell \) is symmetric). Then, there are disjoint open sets \( U_\ell \) and \( U_r \) such that \( \ell \in U_\ell \), \( r \in U_r \), and \( x \not< f y \) for every \( x \in U_\ell \) and \( y \in U_r \), see [4]. Next, take \( \ell_n \in U_\ell \) and \( r_m \in U_r \) such that \( n > m \). Then, we have \( \ell_n \not< f r_m \). However, by (3), (4) and (5), we get that \( \{ r_m, \ell_n \} \in \langle \{ R_m, W_m \} \rangle \cap F_2(X) \), and therefore \( f(\{ r_m, \ell_n \}) = r_m \). This is clearly impossible, so \( \ell = r \).

Having already established this, let us observe that \( b = \ell = r \) implies \( b \in L \cap R \) because \( \ell \in L \) and \( r \in R \). However, \( L \cap R = \{ a \} \) which finally implies that \( b = a \).

We are now ready to prove that, for instance, \( \{ W_n : n < \omega \} \) is a local base at \( a \). To this end, suppose if possible that this fails. Hence, there exists an open neighbourhood \( U \) of \( a \) such that \( W_n \setminus U \neq \emptyset \) for every \( n < \omega \). Next, whenever \( n < \omega \), take a point \( t_n \in W_n \setminus U \). Since \( X \) is compact, \( \{ t_n : n < \omega \} \) has a cluster point \( t \not\in U \). Then, \( t \not< f a \) and, as before, we may find disjoint open sets \( U_t \) and \( U_a \) such that \( t \in U_t \), \( a \in U_a \), and \( x \not< f y \) for every \( x \in U_t \) and \( y \in U_a \). Next, take \( t_n \in U_t \) and \( r_m \in U_a \) such that \( n > m \). Then, \( t_n \not< f r_m \); while, by (4) and (5), \( r_m \not< f t_n \) because \( \{ r_m, t_n \} \in \langle \{ R_m, W_m \} \rangle \cap F_2(X) \). The contradiction so obtained completes the proof. \( \Box \)

6. Proof of Theorem 1.2

In case \( X \) is a topologically well-orderable space, we may use Theorem 1.1.

Suppose that \( F_2(X) \) has a \( \tau_f \)-continuous selection. If \( X \) is compact, then Theorem 1.2 is, in fact, a result of van Mill and Wattel [6]. Let \( X \) be non-compact. By Theorem 2.1, \( X \) is locally compact. Then, by Theorem 3.1,
\( \mathcal{F}_2(\alpha X) \) has a \( \tau_0 \)-continuous selection \( f \) such that \( f^{-1}(\alpha) = \{ \{ \alpha \} \} \). Relying once again on the result of [6], \( \alpha X \) is a linear ordered topological space with respect to some linear order \( \prec \) on \( \alpha X \). It now suffices to show that there exists a compatible (with the topology of \( \alpha X \)) linear order \( \prec \) on \( \alpha X \) such that \( \alpha \) is either the first or the last element of \( \alpha X \), see [2, Lemma 4.1]. We show this following precisely the proof of Theorem 1.1 in [3]. Namely, let

\[
L = \{ x \in \alpha X : x < \alpha \} \quad \text{and} \quad R = \{ x \in \alpha X : \alpha < x \}.
\]

Note that \( L, R \subset \alpha X \setminus \{ \alpha \} = X \) are open subsets of \( \alpha X \). In case one of these sets is also closed, the desired linear order \( \prec \) on \( \alpha X \) can be defined by exchanging the places of \( L \) and \( R \). Namely, by letting for \( x, y \in \alpha X \) that \( x < y \) if and only if

\[
x, y \in L \text{ and } x < y, \quad x, y \in R \text{ and } x < y, \quad x \in R \text{ and } y \in L.
\]

Finally, let us consider the case \( L \cap R = \{ \alpha \} \). Then, \( \alpha \) is a partion of \( \alpha X \). Hence, by Theorem 5.1, \( \alpha X \) is first countable at \( \alpha \). Let \( C[\alpha] \) be the connected component of \( \alpha \) in \( \alpha X \). Since \( f^{-1}(\alpha) = \{ \{ \alpha \} \} \), it now follows from Theorem 4.1 that \( C[\alpha] = \{ \alpha \} \). Indeed, \( C' = C[\alpha] \cap \{ x \in \alpha X : x \leq \alpha \} \) and \( C'' = C[\alpha] \cap \{ x \in \alpha X : x \geq \alpha \} \) are both connected subsets of \( X \) with \( \alpha \in C' \cap (X \setminus C') \) and \( \alpha \in C'' \cap (X \setminus C'') \) (consider that \( X \setminus C' \supset R \) and \( X \setminus C'' \supset L \)), so that \( C' = \{ \alpha \} \) and \( C'' = \{ \alpha \} \), whence also \( C[\alpha] = \{ \alpha \} \).

Then, \( \alpha X \) has a clopen base at \( \alpha \). Indeed, let \( \ell \in L \) and \( r \in R \). Since \( C[\alpha] \) is also the quasi-component of the point \( \alpha \), there are clopen neighbourhoods \( U_{\ell}, U_r \) of \( \alpha \) such that \( \ell \notin U_{\ell} \) and \( r \notin U_r \). Then,

\[
U = \{ x \in U_{\ell} \cap U_r : \ell < x < r \} = \{ x \in U_{\ell} \cap U_r : \ell \leq x \leq r \}
\]

is a clopen neighbourhood of \( \alpha \) with \( U \subset \{ x \in X : \ell < x < r \} \).

That is, \( \alpha X \) has a clopen base at \( \alpha \) and it is first countable at this point. Then, let \( \{ U_n : n < \omega \} \) be a decreasing clopen base at \( \alpha \), with \( U_0 = \alpha X \). Next, for every point \( x \in X \), let \( n(x) = \max\{ n : x \in U_n \} \) and, for convenience, \( n(\alpha) = \omega \). Finally, we may define a linear order \( \prec \) on \( \alpha X \) by putting \( x \prec y \) if and only if

either \( n(x) < n(y) \) or \( n(x) = n(y) \) and \( x < y \).

Since \( \{ U_n : n < \omega \} \) is a decreasing clopen base at \( \alpha \), the order \( \prec \) is compatible with the topology of \( \alpha X \). It is clear that, with respect to \( \prec \), \( \alpha \) is the last element of \( X \). This completes the proof.

**References**


School of Mathematical and Statistical Sciences, Faculty of Science, University of Natal, King George V Avenue, Durban 4041, South Africa

E-mail address: gutev@nu.ac.za