ON TWO-DIMENSIONAL ANALOGUES FOR SHELL-LIKE BODIES

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The paper deals with the question of reduction of the three-dimensional problem of the geometrical and physical nonlinear theory of elasticity to the two-dimensional one, for shell-like elastic bodies.

I. Let \( R \) and \( \dot{R} \) denote the radius-vectors before and after deformation of the bodies \( \Omega \) and \( \dot{\Omega} \) respectively, moreover

\[
\dot{R}(x_1, x_2, x_3) = R(x_1, x_2, x_3) + U(x_1, x_2, x_3),
\]

\[
\partial_j \dot{R} = \dot{R}_j = \partial_j R + \partial_j U = R_j + \partial_j U
\]

where \( U \) is the displacement vector, \((x_1, x_2, x_3)\) are the curvilinear coordinates in the space.

The equation of equilibrium has the form [1]

\[
\frac{1}{\sqrt{\dot{g}}} \frac{\partial}{\partial x_i} \sqrt{\dot{g}} \sigma^i + \dot{\Phi} = 0
\]

where \( \dot{g} \) is the discriminant of the metric tensor of the domain \( \dot{\Omega} \), \( \sigma^i \) are „contravariant stress vectors“, \( \dot{\Phi} \) is an external force.

Under repeating indices summation is meant unless otherwise stated. The Latin letters take the values 1,2,3, while the Greek letters take the values 1,2.

The equilibrium equation can be written as

\[
\frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \sqrt{g} \sigma^i + \Phi = 0, \quad \sigma^i = \hat{\sigma}^{ij}(R_j + \partial_j U),
\]

where \( g \) is the discriminant of the metric tensor of the domain \( \Omega \), \( \sigma^{ij} \) are contravariant components of the stress tensor,

\[
\sigma^i = \sqrt{g} \sigma^i, \quad \Phi = \sqrt{g} \dot{\Phi}.
\]

The stress-strain relation has the form

\[
\sigma^{ij} = (E^{ijmn} + F^{ijmnpq} \varepsilon_{pq}) \varepsilon_{mn},
\]
where $E_{ijmn}$ and $E_{ijmnpq}$ are tensors of elasticity of the fourth and sixth rank, respectively, $\varepsilon_{ij}$ are covariant components of the strain tensor, moreover

$$E_{ijmn} = \lambda g_{ij} g_{mn} + \mu (g_{im} g_{jn} + g_{jn} g_{im}),$$
$$E_{ijmnpq} = \eta_1 g_{ij} g_{mnpq} + \eta_2 g_{ij} (g_{mp} g_{nq} + g_{mq} g_{np}) + \eta_3 g_{mn} (g_{ip} g_{jq} + g_{iq} g_{jp}) + \eta_4 g_{pq} (g_{im} g_{jn} + g_{in} g_{jm}),$$
$$\varepsilon_{mn} = \frac{1}{2} (R_m \partial_n U + R_n \partial_m U + \partial_m U \partial_n U),$$

$\lambda$ and $\mu$ are Lame’s constants of elasticity, and $\eta_1, \eta_2, \eta_3, \eta_4$ are modules of elasticity of the second order for isotropic elastic bodies, $g^{ij}$ are contravariant components of the metric tensor, $R_i$ and $R^i$ are co and contravariant base vectors.

II. 1. We consider the coordinate system of lines of curvature, which is connected normally to the midsurface $S$ of the shell $\Omega$, i.e.

$$R(x_1, x_2, x_3) = r(x_1, x_2) + x_3 n(x_1, x_2),$$

where $r$ and $n$ are radius-vectors and a normal of the $S$, $x_3$ is the thickness coordinate $-h \leq x_3 \leq h$, $h = const$ is the semi-thickness.

The dependence between covariant and contravariant base vectors of the shell $\Omega$ and the midsurface $S$, are expressed as follows

$$R_\alpha = (1 - k_\alpha x_3) r_\alpha, R^\alpha = \frac{r^\alpha}{1 - k_\alpha x_3}, R_3 = R^3 = n, (\alpha = 1, 2)$$

(on $\alpha$ no summation!)

where $k_1$ and $k_2$ are main curvatures of the midsurface $S$, i.e.

$$R_i = (1 - k_i x_3) r_i, \quad R^i = \frac{r^i}{1 - k_i x_3}, \quad g^{ij} = \frac{a^{ij}}{(1 - k_i x_3)(1 - k_j x_3)},$$

$$\sqrt{a} = \sqrt{a} (1 - k_1 x_3)(1 - k_2 x_3), \quad a = a_{11} a_{22} - a_{12}^2,$$

$$a^{ij} = r^i r^j = \begin{cases} a^{\alpha\beta}, & i = \alpha, j = \beta; \\
0, & i = 3, j = \beta, \text{ or } i = \alpha, j = 3; \\
1, & i = j = 3,
\end{cases}$$

$a_{\alpha\beta} = r_\alpha r_\beta, \quad a_{\alpha 3} = a_{3\beta} = 0, \quad a_{33} = 0, \quad k_3 = 0$

(on $i, j, \alpha, \beta$ no summation!).

For the tensor $\varepsilon_{ij}$ we obtain

$$\varepsilon_{ij} = \frac{1}{2} (1 - k_i x_3) (1 - k_j x_3) \left( r_j \frac{\partial_i U}{1 - k_i x_3} + r_i \frac{\partial_j U}{1 - k_j x_3} + \frac{\partial_i U}{1 - k_i x_3} \frac{\partial_j U}{1 - k_j x_3} \right),$$
\[\varepsilon_{\alpha\beta} = \frac{1}{2}(1 - k_\alpha x_3)(1 - k_\beta x_3)(r_\alpha \frac{\partial U}{1 - k_\alpha x_3} + r_\beta \frac{\partial U}{1 - k_\beta x_3} + \frac{\partial U}{1 - k_\alpha x_3} \frac{\partial U}{1 - k_\beta x_3}) = (1 - k_\alpha x_3)(1 - k_\beta x_3)e_{\alpha\beta},\]
\[\varepsilon_{\alpha 3} = \frac{1}{2}(1 - k_\alpha x_3)(n \frac{\partial U}{1 - k_\alpha x_3} + r_\alpha \frac{\partial U}{1 - k_\alpha x_3} + \frac{\partial U}{1 - k_\alpha x_3} \frac{\partial U}{1 - k_\alpha x_3}) = (1 - k_\alpha x_3)e_{\alpha 3},\]
\[\varepsilon_{33} = n\partial_3 U + \frac{1}{2}(\partial_3 U)^2 = e_{33}.\]

2. Now, we assume the validity of the representations:
\[\partial_1 U = (1 - k_1 x_3)\partial_1 V(x_1, x_2),\]
\[\partial_2 U = (1 - k_2 x_3)\partial_2 V(x_1, x_2),\]
\[\partial_3 U = \hat{V}(x_1, x_2),\]
where \(V\) and \(\hat{V}\) are the two-dimensional vectors of \(x_1, x_2\).

Taking into consideration the condition
\[\partial_1 \partial_2 U = \partial_2 \partial_1 U \Rightarrow \partial_1 (k_1 \partial_1 V) = \partial_2 (k_2 \partial_2 V),\]
\[\partial_3 \partial_1 U = \partial_1 \partial_3 U \Rightarrow \partial_1 \hat{V} = -k_1 \partial_1 V,\]
\[\partial_3 \partial_2 U = \partial_2 \partial_3 U \Rightarrow \partial_2 \hat{V} = -k_2 \partial_2 V,\]
for \(V(x_1, x_2)\) we obtain the following equation
\[(k_1 - k_2) \frac{\partial^2 V}{\partial x_1 \partial x_2} + \frac{\partial k_1}{\partial x_2} \frac{\partial V}{\partial x_1} - \frac{\partial k_2}{\partial x_1} \frac{\partial V}{\partial x_2} = 0.\]

Now, from the system of Gauss equations
\[\frac{\partial k_1}{\partial x_2} = (k_2 - k_1) \frac{\partial \ln \sqrt{a_{11}}}{\partial x_2},\]
\[\frac{\partial k_2}{\partial x_1} = (k_1 - k_2) \frac{\partial \ln \sqrt{a_{22}}}{\partial x_1},\]
we have
\[(k_1 - k_2) \left[ \frac{\partial^2 V}{\partial x_1 \partial x_2} - \frac{\partial \ln \sqrt{a_{11}}}{\partial x_2} \frac{\partial V}{\partial x_1} - \frac{\partial \ln \sqrt{a_{22}}}{\partial x_1} \frac{\partial V}{\partial x_2} \right] = 0.\]

The general solution of this equation has the form [2]
\[V(x_1, x_2) = u(x_1) + v(x_2) - \int_{x_1^0}^{x_1} u(t) \frac{\partial R(t, x_2^0, x_1, x_2)}{\partial t} dt - \int_{x_2^0}^{x_2} v(\tau) \frac{\partial R(x_1^0, t, x_1, x_2)}{\partial \tau} d\tau.\]
where $R(t, \tau, x_1, x_2)$ is a Riemann function, $u(x_1)$ and $v(x_2)$ are arbitrary vectors.

For the vector $U(x_1, x_2, x_3)$ we obtain

\[
U(x_1, x_2, x_3) = \int_{x_1^0}^{x_2} [1 - x_3 k_1(x_1, x_2)] \frac{\partial V(x_1, x_2)}{\partial x_1} dx_1 + \\
+ \int_{x_2^0}^{x_2} [1 - x_3 k_2(x_1^0, x_2^0)] \frac{\partial V(x_1^0, x_2^0)}{\partial x_2} dx_2 + \\
+ (x_3 - x_3^0) \hat{V}(x_1^0, x_2^0) + U(x_1^0, x_2^0, x_3^0),
\]

and

\[
\hat{V} = - \int_{x_1^0}^{x_1} k_1(x_1, x_2) \frac{\partial V}{\partial x_1} dx_2 - \int_{x_2^0}^{x_2} k_2(x_1^0, x_2^0) \frac{\partial V(x_1^0, x_2^0)}{\partial x_2} dx_2 + \hat{V}(x_1^0, x_2^0).
\]

Now for $e_{mn}$ we have the following two-dimensional expressions:

\[e_{\alpha\beta} = \frac{1}{2} (r_{\alpha} \partial_{\beta} V + r_{\beta} \partial_{\alpha} V + \partial_{\alpha} V \partial_{\beta} V),\]

\[e_{\alpha3} = \frac{1}{2} (n \partial_{\alpha} V + r_{\alpha} V \hat{V} + \hat{V} \partial_{\alpha} V),\]

\[e_{33} = n \hat{V} + \frac{1}{2} \hat{V}^2.\]

The „contravariant stress vector“ $\sigma^i$ has the form

\[\sigma^i = (1 - k_i x_3) \sigma^{ij}(r_j + \partial_j v) = \frac{T^i}{1 - k_i x_3}\]

(on $i$ no summation!), where

\[T^i = (M^{ijmn} + M^{ijmpq} e_{pq}) e_{mn}(r_j + \partial_j v),\]

\[\partial_j V = \begin{cases} \partial_{\alpha} V, & j = \alpha, \\
\hat{V}, & j = 3. \end{cases}\]

Here

\[M^{ijmn} = \lambda a^{ij} a^{mn} + \mu (a^{im} a^{jn} + a^{in} a^{jm}),\]

\[M^{ijmpq} = \eta a^{ij} a^{mn} a^{pq} + \eta_2 a^{ij} (a^{pm} a^{mq} + a^{mp} a^{nq}) +
+ \eta_3 a^{mn} (a^{ip} a^{jq} + a^{iq} a^{jp}) + \eta_4 a^{pq} (a^{im} a^{jn} + a^{in} a^{jm}).\]

At last, we obtain the following two-dimensional equation of equilibrium:

\[\frac{1}{\sqrt{a}} \left( \frac{\partial \sqrt{a}(1 - k_2 x_3) T^1}{\partial x_1} + \frac{\partial \sqrt{a}(1 - k_1 x_3) T^2}{\partial x_2} \right) + \]
\[ \frac{\partial(1 - k_1 x_3)(1 - k_2 x_3)}{\partial x_3} T^3 + (1 - k_1 x_3)(1 - k_2 x_3) \Phi = 0, \]

where

\[ T^\alpha = (M^{\alpha 3 mn} + M^{\alpha 3 mnpq} e_{pq}) e_{mn}(r_\beta + \partial_\beta V) + \]
\[ + (M^{\alpha 3 mn} + M^{\alpha 3 mnpq} e_{pq}) e_{mn}(n + \hat{V}), \]
\[ T^3 = (M^{3 3 mn} + M^{3 3 mnpq} e_{pq}) e_{mn}(r_\beta + \partial_\beta V) + \]
\[ + (M^{3 3 mn} + M^{3 3 mnpq} e_{pq}) e_{mn}(n + \hat{V}). \]

3. Let us consider the boundary condition for the stresses.

The stress vector \( \sigma_i \) acting onto area with the normal \( \hat{l} \) has the form

\[ \sigma_{i (\hat{l})} = \hat{\sigma}_i \hat{l}_i (\hat{l}_i = \hat{\hat{l}}_i). \]

The normal \( \hat{l} \) after deformation can be defined as

\[ \hat{l} = \frac{\hat{s}_1 \times \hat{s}_2}{|\hat{s}_1 \times \hat{s}_2|}, \]

where \( \hat{s}_1 \) and \( \hat{s}_2 \) are unit tangent vectors of the boundary surface \( \partial \hat{\Omega} \), with the surface element

\[ d\hat{S} = |\hat{s}_1 \times \hat{s}_2| d\hat{s}_1 d\hat{s}_2. \]

Then we have

\[ \hat{l} = \frac{1}{|\hat{s}_1 \times \hat{s}_2|} \left( \frac{d \hat{R}_i}{d \hat{s}_1} \times \frac{d \hat{R}_j}{d \hat{s}_2} \right) = \frac{1}{|\hat{s}_1 \times \hat{s}_2|} \left( \frac{d \hat{R}_i}{d \hat{s}_1} \times \frac{d \hat{R}_j}{d \hat{s}_2} \right) d\hat{s}_1 d\hat{s}_2 = \]
\[ = \hat{R}_i \times \hat{R}_j \frac{d x_i}{d s_1} \frac{d x_j}{d s_2} d\hat{s} \]
\[ = \sqrt{g} \epsilon_{ijk} R^k \frac{d x_i}{d s_1} \frac{d x_j}{d s_2} d\hat{s} \]
\[ = \sqrt{g} (R_1 \times R_2) R_k \frac{d x_i}{d s_1} \frac{d x_j}{d s_2} d\hat{s} \]
\[ = \sqrt{g} (s_1 \times s_2) R_k \frac{d s_k}{d\hat{s}} d\hat{s} \]
\[ = \sqrt{g} (s_1 \times s_2) R_k \frac{d s_k}{d\hat{s}} d\hat{s} \]
\[ \Rightarrow \frac{d \hat{l}_i}{d\hat{s}} = \sqrt{g} (R_k \hat{s}_k d\hat{s}) \Rightarrow \]
\[ \Rightarrow \hat{l}_i = \hat{\hat{l}}_i = \sqrt{g} \frac{d \hat{l}_i}{d\hat{s}} (\hat{l}_i = \hat{R}_i), \]
where \( l = \frac{s_1 \times s_2}{|s_1 \times s_2|} \) is the normal of the boundary surface before deformation, \( dS \) is the element of this surface,

\[
dS = |s_1 \times s_2| ds_1 ds_2,
\]

\( \in_{ijk} \) are the Levi-Civita symbols.

Now the stress vector can be written as

\[
\sigma^* (i) = \sigma l_i = \sqrt{g} \sigma^{ij} l_i \frac{dS}{dS} = \sigma^{ij} l_i \frac{dS}{dS},
\]

i.e.,

\[
\sigma^* \frac{dS}{dS} = \sigma^{ij} l_i = \sigma^\alpha l_\alpha + \sigma^3 l_3,
\]

where

\[
l_\alpha = l R_\alpha, \quad l_3 = ln.
\]

On the surfaces \( x_3 = \pm h \) we have \( l = n \) and so

\[
\sigma(n)(x_1, x_2, \pm h) = \sigma^3(x_1, x_2, \pm h).
\]

The stress vector \( \sigma \), acting on the lateral surface \( d\hat{S} = d\hat{s} dx_3 \) with the normal \( \hat{l} \) has the form

\[
\sigma(i) = \sigma^\alpha (\hat{l} R_\alpha).
\]

The normal \( \hat{l} \) before deformation can be defined as:

\[
\hat{l} = \frac{dR}{d\hat{s}} \times n,
\]

where

\[
\frac{dR}{d\hat{s}} = \hat{s} = \frac{dR}{ds} \frac{ds}{d\hat{s}} = \frac{d(r + x_3 n)}{ds} \frac{ds}{d\hat{s}} = \left( s + x_3 \frac{dn}{ds} \right) \frac{ds}{d\hat{s}} \Rightarrow
\]

\[
\Rightarrow \hat{s} = \left[ (1 - k_3 x_3) s + \tau_s x_3 l \right] \frac{ds}{d\hat{s}}
\]

Therefore,

\[
\hat{l} = \left[ (1 - k_3 k_s) l - x_3 \tau_s s \right] \frac{ds}{d\hat{s}}
\]

\[
(\hat{l} \times \hat{s} = n, \quad l \times s = n),
\]

where \( \hat{l}, \hat{s} \) and \( l, s \) are the unit vectors of the tangential normal and tangent of the lateral curve of the surfaces \( x_3 = \text{const} \) and \( x_3 = 0 \) (midsurface), respectively, \( k_s \) and \( \tau_s \) are the normal curvature and geodesic torsion of the midsurface, \( d\hat{s} \) and \( ds \) are linear elements of the surfaces \( x_3 = \text{const} \) and \( x_3 = 0 \), respectively, moreover

\[
d\hat{s} = \sqrt{1 - 2x_3 k_s + x_3^2 (k_s^2 + \tau_s^2)} ds \Rightarrow
\]
\[ d\hat{s} = \sqrt{a_{11}(1 - k_1x_3)^2 \left( \frac{dx_1}{ds} \right)^2 + a_{22}(1 - k_2x_3)^2 \left( \frac{dx_2}{ds} \right)^2} \, ds. \]

On the other hand, we have [3]
\[ \hat{l} = \frac{dR}{ds} \times n = \frac{dR}{ds} \times n \frac{ds}{d\hat{s}} = R_\alpha \times n \frac{ds}{d\hat{s}} = \sqrt{g} \epsilon_{\alpha \beta} R^\beta \frac{dx_\alpha}{ds} \frac{ds}{d\hat{s}} = \]
\[ = \sqrt{\frac{g}{a}} \epsilon_{\alpha \beta} R^\beta \frac{dx_\alpha}{ds} \frac{ds}{d\hat{s}} = \sqrt{\frac{g}{a}} (r_\alpha \times n) r^\beta R^\beta \frac{ds}{d\hat{s}} \Rightarrow \]
\[ = \frac{dR}{ds} = \frac{ds}{d\hat{s}}. \]

Therefore,
\[ \sigma(\hat{l}) = \sigma^\alpha \hat{l}_\alpha = \sqrt{\frac{g}{a}} \sigma^\alpha \frac{ds}{d\hat{s}} \Rightarrow \]
\[ \Rightarrow \sigma(\hat{l}) \frac{ds}{d\hat{s}} = (1 - k_1x_3)(1 - k_2x_3)(\sigma_1 l_1 + \sigma_2 l_2) \Rightarrow \]
\[ \sigma(\hat{l}) \frac{ds}{d\hat{s}} = (1 - k_2x_3)T_1 l_1 + (1 - k_1x_3)T_2 l_2. \]

Thus, we obtain the following system of two-dimensional equations of the geometrically and physically non-linear theory for shell-like elastic bodies:

a) Equilibrium equations
\[ \frac{1}{\sqrt{a}} \left( \frac{\partial \sqrt{a}(1 - k_2x_3)T^1}{\partial x_1} + \frac{\partial \sqrt{a}(1 - k_1x_3)T^1}{\partial x_2} \right) - 2(H - Kx_3)T^3 + F = 0 \]
\[ \begin{pmatrix} 2H = k_1 + k_2, K = k_1 k_2, (1 - k_1x_3)(1 - k_2x_3)F = (0\ F) + (1\ F) \end{pmatrix}; \]

b) Stress-strain relation
\[ T^i(x_1, x_2) = (M_{i j mn} + M_{i j mnpq} e_{pq}) e_{mn}(r_\beta + \partial_\beta V) + \]
\[ +(M_{i j mn} + M_{i j mnpq} e_{pq}) e_{mn}(n + \hat{V}), \]
where
\[ e_{\alpha \beta} = \frac{1}{2}(r_\alpha \partial_\beta V + r_\beta \partial_\alpha V + \partial_\alpha V \partial_\beta V), \]
\[ e_{\alpha 3} = \frac{1}{2}(n \partial_\alpha V + n \hat{V} + \hat{V} \partial_\alpha V), \]
\[ e_{33} = n \hat{V} + \frac{1}{2} \hat{V}^2. \]

III. Special cases
1. Spherical shell \((k_1 = k_2 = -\frac{1}{R})\)

The vector of displacement \(U\) for the spherical shell has the form

\[
U(x_1, x_2, x_3) = \left(1 + \frac{x_3}{R}\right) V(x_1, x_2) \Rightarrow
\]

\[
\Rightarrow \begin{cases}
\partial_\alpha U = \left(1 + \frac{x_3}{R}\right) \partial_\alpha V \quad (\alpha = 1, 2), \\
\partial_3 U = \frac{1}{R} V.
\end{cases}
\]

The equation of equilibrium can be written as:

\[
\frac{1}{\sqrt{a}} \partial_\alpha T^\alpha + \frac{2}{R} T^3 + F = 0,
\]

where

\[
T^i(x_1, x_2) = (M^{ijmn} + M^{ijmpq} e_{pq}) e_{mn}(r_j + \partial_j V) \quad (\partial_3 V = \frac{1}{R} V),
\]

\[
e_{mn}(x_1, x_2) = \frac{1}{2} (r_m \partial_n V + r_n \partial_m V + \partial_m V \partial_n V),
\]

\[
F(x_1, x_2) = \left(1 + \frac{x_3}{R}\right) \Phi.
\]

The stress vector has the form

\[
\sigma(t) = \left(1 + \frac{x_3}{R}\right) T^\alpha t_\alpha \frac{ds}{d\hat{s}} = T^\alpha t_\alpha = T(t),
\]

\[
(d\hat{s} = \left(1 + \frac{x_3}{R}\right) ds).
\]

2. Cylindrical shell \((k_1 = -\frac{1}{R}, k_2 = 0)\)

The vector of displacement for the cylindrical shell has the form

\[
U(x_1, x_2, x_3) = \left(1 + \frac{x_3}{R}\right) u(x_1) + v(x_2).
\]

Then

\[
\begin{cases}
\partial_1 v = \left(1 + \frac{x_3}{R}\right) \frac{du(x_1)}{dx_1} \\
\partial_2 v = \frac{dv(x_2)}{dx_2} \\
\partial_3 v = \frac{1}{R} u(x_1)
\end{cases}
\]

The equilibrium equation looks like:

\[
\frac{1}{\sqrt{a}} \partial_\alpha T^1 + \frac{1}{R} T^3 = 0,
\]
where

\[ T^i = (M^{i1mn} + M^{i1mnpq}e_{pq})e_{mn}(r_1 + \frac{du(x_1)}{dx_1}) + (M^{i2mn} + M^{i2mnpq}e_{pq})e_{mn}(r_2 + \frac{dv(x_2)}{dx_2}) + (M^{i3mn} + M^{i3mnpq}e_{pq})e_{mn}(n + \frac{1}{R}u(x_1)). \]

Here

\[ M^{ijmn} = \lambda\delta^{ij}\delta^{mn} + \mu(\delta^{im}\delta^{jn} + \delta^{in}\delta^{jm}), \]
\[ M^{ijmnpq} = \eta_1\delta^{ij}\delta^{mn}\delta^{pq} + \eta_2\delta^{ij}(\delta^{mn}\delta^{op} + \delta^{op}\delta^{mn}) + \eta_3\delta^{mn}(\delta^{ip}\delta^{jq} + \delta^{iq}\delta^{jp}) + \eta_4\delta^{pq}(\delta^{im}\delta^{jn} + \delta^{in}\delta^{jm}), \]
\[ \left( \delta^{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \right). \]

For the \( e_{ij} \) we obtain

\[ e_{12} = \frac{1}{2} \left( r_1 \frac{du(x_1)}{dx_1} + r_2 \frac{dv(x_2)}{dx_2} + \frac{du}{dx_1} \frac{dv}{dx_2} \right) = e_{21}, \]
\[ e_{13} = \frac{1}{2} \left( n \frac{du}{dx_1} + \frac{r_1 u}{R} + \frac{u}{R} \frac{du}{dx_1} \right), \]
\[ e_{23} = \frac{1}{2} \left( n \frac{dv}{dx_2} + \frac{r_2 u}{R} + \frac{u}{R} \frac{dv}{dx_2} \right), \]
\[ e_{11} = r_1 \frac{du}{dx_1} + \frac{1}{2} \left( \frac{du}{dx_1} \right)^2, \quad e_{22} = r_2 \frac{dv}{dx_2} + \frac{1}{2} \left( \frac{dv}{dx_2} \right)^2, \quad e_{33} = \frac{nu}{R} + \frac{u^2}{2R^2}. \]

The stress vector \( \sigma(\hat{l}) \) has the form \( (\hat{l} = r_2) \):

\[ \sigma(\hat{l}) = \sigma(r_2) = (1 + \frac{x_3}{R})T^2l_2 \frac{ds}{d\hat{s}} = T^2. \]

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