A COMMUTATOR THEOREM FOR FRACTIONAL INTEGRALS IN SPACES OF HOMOGENEOUS TYPE

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Abstract. We give a new proof of a commutator theorem for fractional integrals in spaces of homogeneous type.

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1. Introduction. Bramanti and Cerutti [3] and Bramanti [2] extended a classical commutator theorem for fractional integrals due to Chanillo [5] to the context of spaces of homogeneous type. In [3] Bramanti and Cerutti follow an idea contained in [7], based in holomorphic families of operators, used to study the $L^p$ boundedness of singular integrals in Euclidean spaces. In [2] Bramanti investigated the boundedness of the commutator of certain integral operators having positive kernels. A fractional integral appears as a particular case. Bramanti deduces the boundedness of the commutator from a suitable inequality that involves the maximal sharp function. In this paper, we give a different proof to the commutator theorem for fractional integrals in spaces of homogeneous type. We follow the original proof of Chanillo [5] and a good $\lambda$ inequality is essential.

We firstly recall the main definitions needed in the paper (see [8, 9, 11]). $(X, \delta, \mu)$ will be a space of homogeneous type. That is, $X$ is a nonvoid set, $\delta$ is a quasidistance on $X$, i.e., $\delta: X \times X \to [0, \infty)$ is a function satisfying the following properties:

(i) $\delta(x, y) = 0$ if and only if $x = y$,
(ii) $\delta(x, y) = \delta(y, x)$, for every $x, y \in X$, and
(iii) there exists a positive constant $k$ such that for every $x, y, z \in X$

$$\delta(x, y) \leq k(\delta(x, z) + \delta(z, y)),$$

and $\mu$ is a positive regular measure on $X$ defined on a $\sigma$-algebra of subsets of $X$ which contains the open sets (in the topology induced by the uniform structure associated to $\delta$) and the ball $B(x, r) = \{y \in X : \delta(x, y) < r\}$, for every $x \in X$ and $r > 0$, and that satisfies the doubling condition: there exists $A > 0$ for which

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)),$$

for each $x \in X$ and $r > 0$. Note that if $X$ has more than one element, then $k \geq 1$. The trivial case $k < 1$ is not considered in this paper.

There are many interesting examples of spaces of homogeneous type. For instance, any $C^\infty$ compact Riemannian manifold with the Riemannian metric and volume and
the boundary of any bounded Lipschitz domain in $\mathbb{R}^n$ with the induced Euclidean metric and the Lebesgue measure are spaces of homogeneous type.

A space of homogeneous type is said to be normal if there exist positive constants $A_1$ and $A_2$ such that for every $x \in X$,

$$A_1 r \leq \mu(B(x,r)), \quad \text{when } 0 < r < R_x,$$

$$\mu(B(x,r)) \leq A_2 r, \quad \text{if } r \geq r_x,$$

where

$$R_x = \begin{cases} \infty, & \text{if } \mu(X) = \infty, \\ \inf \{ r > 0 : B(x,r) = X \}, & \text{if } \mu(X) < \infty, \end{cases}$$

$$r_x = \begin{cases} 0, & \text{if } \mu(\{x\}) = 0, \\ \sup \{ r > 0 : B(x,r) = \{x\} \}, & \text{if } \mu(\{x\}) > 0. \end{cases}$$

Sufficient conditions, in order that a space $(X,\delta,\mu)$ of homogeneous type admits a quasidistance $d$ that is equivalent to $\delta$ and such that $(X,d,\mu)$ is normal, are given in [14, Lemma 22].

A space of homogeneous type is of order $\rho$, $0 < \rho \leq 1$, if there is a positive constant $C$ such that for every $x,y,z \in X$

$$|\delta(x,z) - \delta(y,z)| \leq C \delta(x,y)^\rho \left( \max \{ \delta(x,z), \delta(y,z) \} \right)^{1-\rho}. \quad (1.5)$$

For each $1 \leq p \leq \infty$, $L^p(X,\mu)$ and $\| \cdot \|_p$ have the usual meanings. We say that a complex valued measurable function $f$ on $X$ is in $L^p_{\text{loc}}(X,\mu)$, $1 \leq p < \infty$, if $\int_{B(x,r)} |f(x)|^p d\mu(x) < \infty$, for every $r > 0$ and for some (and then for all) $x \in X$.

Let $b \in L^1_{\text{loc}}(X,\mu)$. We define $b_\epsilon(x)$, with $x \in X$ and $\epsilon > 0$, as the mean value

$$b_\epsilon(x) = \frac{1}{\mu(B(x,\epsilon))} \int_{B(x,\epsilon)} b(y) d\mu(y). \quad (1.6)$$

If $1 \leq p < \infty$ we will say that a function $b \in L^p_{\text{loc}}(X,\mu)$ is in $\text{BMO}_p$ if and only if,

$$\|b\|_{*p} = \sup_{\epsilon > 0} \left\{ \frac{1}{\mu(B(x,\epsilon))} \int_{B(x,\epsilon)} \{ b(y) - b_\epsilon(x) \}^p d\mu(y) \right\}^{1/p} < \infty. \quad (1.7)$$

We define on $\text{BMO}_p$ a “norm” as follows:

$$\|b\|^{(p)} = \begin{cases} \|b\|_{*p}, & \text{if } \mu(X) = \infty, \\ \|b\|_{*p} + \left| \int_X b(x) d\mu(x) \right|, & \text{if } \mu(X) < \infty. \end{cases} \quad (1.8)$$

When $\mu(X) < \infty$, $(\text{BMO}_p, \| \cdot \|^{(p)})$ is a Banach space. If $\mu(X) = \infty$, then we introduce in $\text{BMO}_p$ the following relation: let $b_1$ and $b_2$ be in $\text{BMO}_p$,

$$b_1 \sim b_2 \iff \text{there exists } C \in \mathbb{C} \text{ such that } b_1 - b_2 = C. \quad (1.9)$$

It is clear that if $b_1, b_2 \in \text{BMO}_p$ and $b_1 \sim b_2$, then $\|b_1\|^{(p)} = \|b_2\|^{(p)}$. The quotient space $\text{BMO}_p / \sim$ will be denoted again by $\text{BMO}_p$ and by considering on it the norm
induced by $\| \cdot \|^{(p)}$, $\text{BMO}_p$ is a Banach space. As it was proved by Coifman and Weiss [9, page 594], if $1 \leq p, q < \infty$, the spaces $\text{BMO}_p$ and $\text{BMO}_q$ coincide and the norms $\| \cdot \|^{(p)}$ and $\| \cdot \|^{(q)}$ are equivalent. In the sequel, as usual, we will denote by $\text{BMO}$ the space $\text{BMO}_p$, $1 \leq p < \infty$.

Let $0 \leq \alpha < 1$. The fractional maximal function $M_\alpha f$ of $f \in L^1_{\text{loc}}(X, \mu)$ is defined by

$$(M_\alpha f)(x) = \sup_{B : x \in B} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)| \, d\mu(y), \quad x \in X. \quad (1.10)$$

Here, for each $x \in X$, the supremum is taken over all those $B$ balls in $X$ containing to $x$. As usual we denote by $M$ the maximal operator $M_0$.

The fractional integral of order $\alpha$ of $f$, $I_\alpha f$, is given by

$$(I_\alpha f)(x) = \int_{X - \{x\}} \frac{f(y)}{\delta(x, y)^{1-\alpha}} \, d\mu(y). \quad (1.11)$$

In this paper, we study the boundedness of the commutator $[I_\alpha, b]$ of the fractional integral $I_\alpha$ and the multiplier operator associated to a measurable function $b$ on $X$ defined through

$$[I_\alpha, b](f) = bI_\alpha(f) - I_\alpha(bf). \quad (1.12)$$

Throughout this paper, for every $1 \leq p < \infty$, we will denote by $p'$ the conjugate of $p$. By $C$ we will always represent a positive constant not necessarily the same in each occurrence.

The following theorem is the main result of the paper.

**Theorem 1.1.** Let $0 < \alpha < 1$, $0 \leq p < 1$, $1 < p < 1/\alpha$, and $1/q = 1/p - \alpha$. Assume that $(X, \delta, \mu)$ is a normal space of homogeneous type of order $\rho$ such that $\mu(\{x\}) = 0$, $x \in X$. Then the commutator operator $[I_\alpha, b]$ is bounded from $L^p(X, \mu)$ into $L^q(X, \mu)$ provided that $b \in \text{BMO}$.

Let now $(X, \delta, \mu)$ be a normal space of homogeneous type and of order $\rho \in (0, 1)$, such that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$, for every $x \in X$. Gatto, Segovia, and Vagi [10] defined, for every $0 < \alpha < 1$, a function $\delta_\alpha$ on $X \times X$ as follows:

$$\delta_\alpha(x, y) = \left( \int_0^\infty t^{\alpha-1} s(x, y, t) \, dt \right)^{1/\alpha-1}, \quad \text{for } x \neq y, \quad (1.13)$$

where $s$ represents a symmetric approximation to the identity in the sense of Coifman, and

$$\delta_\alpha(x, y) = 0, \quad \text{for } x = y. \quad (1.14)$$

In [10, Lemma 2] it is proved that, for every $0 < \alpha < 1$, $\delta_\alpha$ is a quasidistance equivalent to $\delta$. Moreover, for each $0 < \alpha < 1$, $(X, \delta_\alpha, \mu)$ is a normal space of homogeneous type of order $\rho$.

Also these authors introduced the fractional integral $\tilde{I}_\alpha$ of order $\alpha \in (0, 1)$ through

$$(\tilde{I}_\alpha f)(x) = \int_{X - \{x\}} \frac{f(y)}{\delta_\alpha(x, y)^{1-\alpha}} \, d\mu(y). \quad (1.15)$$
If we represent by $BMO(\alpha)$ the $BMO$-space associated to the quasidistance $\delta_\alpha$, $0 < \alpha < 1$, it is immediately deduced from Theorem 1.1 the following commutator theorem for the fractional integral $I_\alpha$.

**Corollary 1.2.** Assume that $(X, \delta, \mu)$ is a normal space of homogeneous type and of order $\rho \in (0, 1)$, such that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$, for every $x \in X$. Let $0 < \alpha < 1$. Then the commutator operator $[I_\alpha, b]$ defined by

$$[I_\alpha, b](f) = bI_\alpha(f) - I_\alpha(bf),$$

is a bounded operator from $L^p(X, \mu)$ into $L^q(X, \mu)$ provided that $1 < p < 1/\alpha$, $1/q = 1/p - \alpha$ and $b \in BMO(\alpha)$.

### 2. The proof of the commutator theorem.

In this section, we will prove Theorem 1.1. To see that result we previously establish six lemmas.

Boundedness of the fractional integral $I_\alpha$ was studied in [11, Theorem 1] and [12, Theorems 2.2 and 2.4].

**Lemma 2.1** (see [11, Theorem 1]). Let $1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$. If $(X, \delta, \mu)$ is a normal space of homogeneous type, then

(i) $I_\alpha$ maps continuously $L^p(X, \mu)$ into $L^q(X, \mu)$.

(ii) There exists $C_1 > 0$ such that

$$\mu(\{x \in X : |I_\alpha(f)(x)| > \lambda\}) \leq C_1 \left(\frac{\|f\|_1}{\lambda}\right)^{1/1-\alpha},$$

for every $f \in L^1(X, \mu)$ and $\lambda > 0$.

Kokilashvili and Kufner [12, Theorem 3.2] proved a weighted version of [11, Theorem 1].


The following result can be easily inferred from [15, Theorem 4] (also from [12, Proposition A]).

**Lemma 2.2.** Let $1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$. Then $M_\alpha$ is a bounded operator from $L^p(X, \mu)$ into $L^q(X, \mu)$.

We now define the auxiliary operator $C(b, f)$ on $X$ as follows:

$$C(b, f)(x) = \sup_{\epsilon > 0} \left| \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x, y)^{1-\alpha}} f(y) d\mu(y) \right|, \quad x \in X,$$

where $b$ and $f$ are measurable complex functions on $X$.

Next a useful weak type inequality for the operator $C(b, f)$ is established.
**Lemma 2.3.** Assume that \((X, \delta, \mu)\) is a normal space of homogeneous type. Let \(1 < p < 1/\alpha\). If \(f \in L^p(X, \mu)\) and \(b \in L^{p'}(X, \mu)\), then

\[
\mu(\{x \in X : C(b, f)(x) > \lambda\}) \leq C_0 \left( \frac{\|b\|_{p'} \|f\|_p}{\lambda} \right)^{1/1-\alpha}, \text{ for every } \lambda > 0. \tag{2.3}
\]

**Proof.** It is not hard to see that

\[
C(b, f)(x) \leq \sup_{\epsilon > 0} \int_{X \setminus B(x, \epsilon)} \frac{|b(y)| \|f(y)\|}{\delta(x, y)^{1-\alpha}} d\mu(y) + \sup_{\epsilon > 0} |b_{\epsilon}(x)| \int_{X \setminus B(x, \epsilon)} \frac{|f(y)|}{\delta(x, y)^{1-\alpha}} d\mu(y) \leq I_{\alpha}(|bf|)(x) + I_{\alpha}(|f|) M(b)(x), \quad x \in X. \tag{2.4}
\]

Moreover, Holder inequality and Lemmas 2.1 and 2.2 lead to

\[
\int_X M(b)(x)^{1/1-\alpha} I_{\alpha}(|f|)(x)^{1/1-\alpha} d\mu(x)
\leq \left( \int_X M(b)(x)^{p'} d\mu(x) \right)^{1/p'(1-\alpha)} \left( \int_X I_{\alpha}(|f|)(x)^{p'/p'(1-\alpha)-1} d\mu(x) \right)^{1/(1-p'(1-\alpha))}
\leq C \|b\|_{p'}^{1/1-\alpha} \|f\|_p^{1/1-\alpha}. \tag{2.5}
\]

Hence if \(\lambda > 0\), then

\[
\mu(\{x \in X : M(b)(x) I_{\alpha}(|f|)(x) > \lambda\}) \leq C \left( \frac{\|b\|_{p'} \|f\|_p}{\lambda} \right)^{1/1-\alpha}. \tag{2.6}
\]

Also by taking into account Lemma 2.1 we have

\[
\mu(\{x \in X : I_{\alpha}(|bf|)(x) > \lambda\}) \leq C \left( \frac{\|bf\|_1}{\lambda} \right)^{1/1-\alpha} \leq C \left( \frac{\|b\|_{p'} \|f\|_p}{\lambda} \right)^{1/1-\alpha}, \quad \lambda > 0. \tag{2.7}
\]

Now to finish the proof of this lemma it is sufficient to combine (2.4), (2.6), and (2.7).

**Lemma 2.4.** Assume that \((X, \delta, \mu)\) is a normal space of homogeneous type such that \(\mu(\{x\}) = 0, x \in X\). Let \(0 < \alpha < 1\), \(1 < p < 1/\alpha\), \(0 < \beta < 1/k\), and \(d, \gamma > 0\). Let \(b \in \text{BMO}\) and \(f\) be a measurable function. Then

\[
d\gamma \int_{X \setminus B(x_0, d)} \frac{|b(y) - b_d(x_0)|}{\delta(x, y)^{1+y-\alpha}} |f(y)| d\mu(y) \leq C(M_{\alpha p}(|f|^{p'}(x_0))^{1/p} \|b\|_{p', p'}, \tag{2.8}
\]

provided that \(\delta(x, x_0) \leq \beta d\). Here \(C\) is a constant that does not depend on \(d\).
Proof. Suppose that $\mu(X) = \infty$. If $\mu(X) < \infty$ we can proceed in a similar way. Holder inequality implies that

$$
\int_{X \setminus B(x,d)} \frac{|b(y) - b_d(x)|}{\delta(x,y)^{1+\gamma}} |f(y)| \, d\mu(y) 
\leq \left( \int_{X \setminus B(x,d)} \frac{|b(y) - b_d(x)|^{p'}}{\delta(x,y)^{1+\gamma}} \, d\mu(y) \right)^{1/p'}
\times \left( \int_{X \setminus B(x,d)} \frac{|f(y)|^p}{\delta(x,y)^{1+\gamma}} \, d\mu(y) \right)^{1/p}.
$$

(2.9)

Since $\mu$ is doubling we can write for every $x \in X$ and $j \in \mathbb{N}$,

$$
|b_{2^j d}(x) - b_{2^{j+1} d}(x)| \leq \frac{1}{\mu(B(x,2^{j-1} d))} \int_{B(x,2^{j-1} d)} |b(y) - b_{2^j d}(x)| \, d\mu(y) 
\leq C \frac{1}{\mu(B(x,2^j d))} \int_{B(x,2^j d)} |b(y) - b_{2^j d}(x)| \, d\mu(y)
\leq C \|b\|_{s,1}.
$$

(2.10)

Hence, it concludes that

$$
|b_d(x) - b_{2^j d}(x)| \leq C_j \|b\|_{s,1}, \quad j \in \mathbb{N}, \ x \in X.
$$

(2.11)

Then, since $(X,\delta,\mu)$ is normal, it follows

$$
\int_{X \setminus B(x,d)} \frac{|b(y) - b_d(x)|^{p'}}{\delta(x,y)^{1+\gamma}} \, d\mu(y) 
\leq \int_{X \setminus B(x,d)} \frac{|b(y) - b_d(x)|^{p'}}{\delta(x,y)^{1+\gamma}} \, d\mu(y) 
\leq C^{2 \gamma} \left( \int_{B(x,2^{j+1} d)} |b(y) - b_{2^j d}(x)|^{p'} \, d\mu(y) \right) 
\leq C \sum_{j=0}^{\infty} \left( \int_{B(x,2^j d)} |b(y) - b_{2^j d}(x)|^{p'} \, d\mu(y) \right) 
\leq C \sum_{j=0}^{\infty} \left( \int_{B(x,2^{j+1} d)} |b(y) - b_{2^j d}(x)|^{p'} \, d\mu(y) \right)
\leq C \sum_{j=0}^{\infty} \frac{2^{-j \gamma}}{2^j d} \left( \frac{1}{\mu(B(x,2^{j+1} d))} \int_{B(x,2^{j+1} d)} |b(y) - b_{2^{j+1} d}(x)|^{p'} \, d\mu(y) \right) 
\leq C \sum_{j=0}^{\infty} \frac{2^{-j \gamma}}{2^j d} \left( \frac{1}{\mu(B(x,2^{j+1} d))} \int_{B(x,2^{j+1} d)} |b(y) - b_{2^j d}(x)|^{p'} \, d\mu(y) \right)
\leq C \sum_{j=0}^{\infty} \frac{2^{-j \gamma}}{2^j d} \leq \|b\|^{p'}_{s,p'}.
$$

(2.12)

On the other hand, if $\delta(x_0, y) \leq \beta d$ and $\delta(x, y) \leq d$, then $\delta(x_0, y) \geq (1 - k\beta) / k) d$ and $\delta(x_0, y) \leq k(\beta + 1) \delta(x, y)$. Hence, by invoking again the normality of $(X,\delta,\mu)$ we can write
\[
d^y \int_{X \setminus B(x, d)} \frac{|f(y)|^p}{\delta(x, y)^{1+y-p\alpha}} d\mu(y)
\]
\[
\leq C d^y \int_{X \setminus B(x_0, (1-k\beta)/kd)} \frac{|f(y)|^p}{\delta(x_0, y)^{1+y-p\alpha}} d\mu(y)
\]
\[
\leq C d^y \sum_{j=0}^{\infty} \int_{2^{j+1}((1-k\beta)/kd) \delta(x_0, y) \geq 2^j((1-k\beta)/kd)} \frac{|f(y)|^p}{\delta(x_0, y)^{1+y-p\alpha}} d\mu(y)
\]
\[
\leq C d^y \sum_{j=0}^{\infty} (d2^j)^{-1-y+p\alpha} \int_{B(x_0, 2^{j+1}((1-k\beta)/kd))} |f(y)|^p d\mu(y)
\]
\[
\leq C \sum_{j=0}^{\infty} 2^{-jy} \frac{1}{\mu(B(x_0, 2^{j+1}((1-k\beta)/kd)))^{1-p\alpha}} \int_{B(x_0, 2^{j+1}((1-k\beta)/kd))} |f(y)|^p d\mu(y)
\]
\[
\leq CM_{p\alpha}(|f|^p)(x_0).
\] (2.13)

Thus the result is proved.

The following Whitney type covering lemma will be useful in the sequel.

**Lemma 2.5** (see [4, Lemma 1] and [13, Lemma 2]). Let \( \Omega \) be a proper open subset of \( X \) and let \( B \) be a ball in \( X \) such that \( B \cap \Omega \neq \emptyset \) and \( B \cap (X \setminus \Omega) \neq \emptyset \). Then there exists a sequence \( (B_j)_{j \in \mathbb{N}} \) of balls in \( X \) satisfying the following three properties:

(i) \( \Omega \cap B \subset \bigcup_{j \in \mathbb{N}} B_j \subset \Omega \cap (B^*)^c \),

(ii) \( B_j^c \cap (X \setminus \Omega) = \emptyset \), \( j \in \mathbb{N} \), and

(iii) \( \mu(\Omega \cap B) \leq \sum_{j=1}^{\infty} \mu(B_j) \leq C \mu(\Omega \cap (B^*)^c) \).

Here if \( B = B(x, r) \), with \( x \in X \) and \( r > 0 \), \( B^* \) denotes the ball \( B(x, r k(2k+1)) \).

Next we will prove in the main lemma a good-\( \lambda \) inequality.

**Lemma 2.6.** Let \( 0 \leq \rho < 1 \) and \( 1 < p < 1/\alpha \). Assume that \( (X, \delta, \mu) \) is a normal space of homogeneous type that is of order \( \rho \) and such that \( \mu([x]) = 0 \), \( x \in X \). Let \( b \in \text{BMO} \) and \( f \) be a measurable function on \( X \). Then there exists \( \beta_0 \) such that for every \( \beta \geq \beta_0 \) and \( \gamma > 1 \)

\[
\mu\left\{ x \in X : C(b, f)(x) > \beta \lambda, \|b\|_{*,p'} \left( I_{\alpha}(|f|)(x) + (M_{p\alpha}(|f|^p)(x))^{1/p} \right) \leq \gamma \lambda \right\}
\]

\[
\leq C \gamma \mu\left\{ x \in X : C(b, f)(x) > \lambda \right\},
\] (2.14)

provided that one of the following two conditions holds:

(i) \( \lambda > 0 \) and \( \mu(X) = \infty \),

(ii) \( \lambda > (C_0/\mu(X))^{1-\alpha} \|b\|_{p'} \|f\|_p \) and \( \mu(X) < \infty \), where \( C_0 \) is the positive constant appearing in Lemma 2.3.

**Proof.** Let \( \beta, \gamma > 0 \) and \( \lambda \) satisfying the imposed conditions. We define the following sets:

\[
E_\lambda(\beta, \gamma) = \left\{ x \in X : C(b, f)(x) > \beta \gamma, \|b\|_{*,p'} \left( I_{\alpha}(|f|)(x) + (M_{p\alpha}(|f|^p)(x))^{1/p} \right) \leq \gamma \lambda \right\},
\] (2.15)

\[
W_\lambda = \left\{ x \in X : C(b, f)(x) > \lambda \right\}.
\]
Note that we can assume, without loss of generality, that \( W_\lambda = \emptyset \) and \( W_\lambda \neq X \). Indeed, suppose firstly that \( \mu(X) = \infty \). If \( W_\lambda = X \), then (2.14) is clear for every \( \beta > 0 \) and \( y > 0 \). On the other hand, if \( \mu(X) < \infty \), then Lemma 2.3 implies that \( \mu(W_\lambda) < \mu(X) \) when \( \lambda > (C_0/\mu(X))^{-1-\alpha}\|b\|_p\|f\|_p \) and where \( C_0 \) is the positive constant that appears in Lemma 2.3. Also if \( \mu(X) \leq \infty \) and \( W_\lambda = \emptyset \), then (2.14) holds for every \( \beta > 1 \) and \( y > 0 \).

Let \( B \) be a ball in \( X \) such that \( B \cap W_\lambda \neq \emptyset \) and \( B \cap (X \setminus W_\lambda) \neq \emptyset \). Then there exists a sequence \((B_j)_{j=1}^\infty\) of balls in \( X \) satisfying conditions (i), (ii), and (iii) in Lemma 2.5 by replacing \( \Omega \) by \( W_\lambda \).

Let \( j \in \mathbb{N} \). Write \( B_j = B(a,d) \), with \( a \in X \) and \( d > 0 \). We define \( B_j^1 = B(a,\alpha_1 d) \) and \( B_j^2 = B(a,\alpha_2 d) \), where \( \alpha_1 \leq k(2k^2(1 + k(2k + 1)) + 1) \) and \( \alpha_2 > k(1 + k(\alpha_1 + 1)) \).

Assume that \( B_j \cap E(\beta,\gamma) \neq \emptyset \) and choose \( x_1 \in B_j \cap E(\beta,\gamma) \). We write \( f = f_1 + f_2 \), where \( f_1 = f|_{B_j^2} \), and \( b = b_1 + b_2 \), being \( b_1 = (b - b_{B_j^2})1_{B_j^2} \) and \( b_{B_j^2} = 1/\mu(B_j^2) \times \int_{B_j^2} b(y) d\mu(y) \).

We have that \( C(b,f_1)(x) \leq C(b_1,f_1)(x) \), for every \( x \in B_j \). Indeed, let \( x \in B_j \) and \( \epsilon > 0 \). Since \( \alpha_2 > k(1 + k(\alpha_1 + 1)) \), if \( B(x,\epsilon) \cap (X \setminus B_j^2) \neq \emptyset \), then \( B_j^1 \subset B(x,\epsilon) \). Hence we can write

\[
(b_1)_\epsilon(x) = \frac{1}{\mu(B(x,\epsilon))} \int_{B(x,\epsilon)} b_1(y) d\mu(y) = \frac{1}{\mu(B(x,\epsilon))} \int_{B(x,\epsilon) \cap B_j^2} (b(y) - b_{B_j^2}) d\mu(y) = \frac{1}{\mu(B(x,\epsilon))} \int_{B(x,\epsilon)} b(y) d\mu(y) - b_{B_j^2},
\]

provided that \( B_j^1 \cap (X \setminus B(x,\epsilon)) \neq \emptyset \).

Then, since \( B_j^1 \subset B_j^2 \), one has

\[
C(b,f_1)(x) = \sup_{\epsilon > 0} \left| \int_{X \setminus B(x,\epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f_1(y) d\mu(y) \right| = \sup_{\epsilon > 0} \left| \int_{X \setminus B(x,\epsilon)} \frac{b_1(y) + b_{B_j^2} - b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f_1(y) d\mu(y) \right| \leq \sup_{\epsilon > 0} \int_{X \setminus B(x,\epsilon)} \frac{b_1(y) - (b_1)_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f_1(y) d\mu(y) = C(b_1,f_1)(x).
\]

Moreover from Lemma 2.3 we deduce that for every \( \beta > 1 \),

\[
\mu(\{ x \in B_j : C(b_1,f_1)(x) > \beta \lambda \}) \leq C \left( \frac{\|b_1\|_{p'}\|f_1\|_p}{\beta \lambda} \right)^{1-\alpha}
\]

\[
= C \left( \int_{B_j} \|b(y) - b_{B_j^2}\|_{p'} d\mu(y) \right)^{1/p'(1-\alpha)} \left( \int_{B_j} |f(y)|^p d\mu(y) \right)^{1/p(1-\alpha)} \leq C \lambda^{1/\alpha-1} \mu(B_j) \left( \|b\|_{*,p'} (M_{p\alpha}(|f|^p)(x_1))^{1/p} \right)^{1/1-\alpha},
\]

because \( \mu \) is doubling.
Hence, since $x_1 \in B_j \cap E$ if $\gamma < 1$, then

$$\mu \{ x \in B_j : C(b, f_1) (x) > \beta \lambda \} \leq C \gamma \mu (B_j).$$

(2.19)

By virtue of (ii) in Lemma 2.5, $B_j^* \cap (X \setminus W_\lambda) \neq \emptyset$. Choose $x_0 \in B_j^* \cap (X \setminus W_\lambda)$, that is, $x_0 \in B_j^*$ and $C(b, f)(x_0) \leq \lambda$.

Now our objective is to estimate

$$\mu \{ x \in B_j : C(b, f_2) (x) > \beta \lambda \}.$$

(2.20)

We consider two cases.

Assume firstly that $\epsilon > \sigma d$, where $\alpha_2/k - 1 > \sigma > (\alpha_1 + 1)k$. Since $\sigma > (\alpha_1 + 1)k$, for every $x \in B_j$, $B^*_j \subset B(x, \epsilon)$. Let $x \in B_j$. We have

$$\int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon (x)}{\delta(x,y)} f_2(y) d\mu (y) = \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon (x)}{\delta(x,y)} f(y) d\mu (y)$$

$$= I_1 + I_2 + I_3,$$

(2.21)

where

$$I_1 = \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon (x)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu (y) - \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon (x_0)}{\delta(x,y)^{1-\alpha}} f(y) d\mu (y),$$

$$I_2 = \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon (x_0)}{\delta(x,y)^{1-\alpha}} f(y) d\mu (y) - \int_{X \setminus B(x_0, \epsilon)} \frac{b(y) - b_\epsilon (x_0)}{\delta(x_0,y)^{1-\alpha}} f(y) d\mu (y),$$

$$I_3 = \int_{X \setminus B(x_0, \epsilon)} \frac{b(y) - b_\epsilon (x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y) d\mu (y).$$

(2.22)

We are going to estimate $I_i$, $i = 1, 2, 3$.

As mentioned above if $\delta(x,y) > \epsilon$, then $y \notin B^*_j$. Hence $\delta(x,y) > \epsilon$ implies that $\delta(x,y) \geq ((\alpha_1 - k)/k) \epsilon > 0$. Then we can write

$$\frac{\delta(x,y)}{\delta(x_1, y)} \leq k (\delta(x,x_1) + \delta(x,y)) \leq \frac{2k^3}{\alpha_1 - k} + k,$$

(2.23)

provided that $\delta(x,y) > \epsilon$.

Therefore it follows

$$|I_1| = \left| \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon (x)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu (y) - \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon (x_0)}{\delta(x,y)^{1-\alpha}} f(y) d\mu (y) \right|$$

$$\leq \int_{X \setminus B(x, \epsilon)} \left| \frac{b_\epsilon (x_0) - b_\epsilon (x)}{\delta(x,y)^{1-\alpha}} \right| f(y) d\mu (y)$$

$$\leq C \left| b_\epsilon (x_0) - b_\epsilon (x) \right| \int_{X \setminus B(x_0, \epsilon)} \left| \frac{f(y)}{\delta(x_1, y)^{1-\alpha}} \right| d\mu (y)$$

$$\leq C \left| b_\epsilon (x_0) - b_\epsilon (x) \right| \left| I_{\alpha} (|f|) (x_1) \right|.$$

(2.24)

Moreover if $y \in B(x_0, \epsilon)$, then

$$\delta(x,y) \leq k (\delta(y,x_0) + \delta(x_0,x)) \leq k (\epsilon + \epsilon (1 + k (2k + 1))) < 2^m \epsilon,$$

(2.25)
where \( m \in \mathbb{N} \) is large enough and \( m \) is not depending on \( d \) and \( \epsilon \).

Hence, since \((X,\delta,\mu)\) is normal we have that

\[
|b_\epsilon(x_0) - b_\epsilon(x)| \leq \frac{1}{\mu(B(x_0,\epsilon))} \int_{B(x_0,\epsilon)} |b(y) - b_\epsilon(x)| d\mu(y)
\]

\[
\leq C \left( \frac{1}{\mu(B(x,2^m\epsilon))} \int_{B(x,2^m\epsilon)} |b(y) - b_\epsilon(x)| d\mu(y) \right)
\]

\[
\leq C \left( \frac{1}{\mu(B(x,2^m\epsilon))} \int_{B(x,2^m\epsilon)} |b(y) - b_{2^m\epsilon}(x)| d\mu(y) + |b_{2^m\epsilon}(x) - b_\epsilon(x)| \right)
\]

\[
\leq C \|b\|_{*,p}'.
\]  

Thus we conclude that

\[
|I_1| \leq C \|b\|_{*,p'} I_\alpha(|f|)(x_1) \leq Cy\lambda.
\]  

(2.27)

On the other hand, to estimate \( I_2 \) we will use that \((X,\delta,\mu)\) is a space of homogeneous type which is of order \( \rho \in (0,1) \). It is clear that

\[
|I_2| \leq \int_{\delta(x,y) \geq \epsilon \text{ and } \delta(x_0,y) \geq \epsilon} |b(y) - b_\epsilon(x_0)| \cdot |f_2(y)| \cdot |\delta(x,y)^{\alpha-1} - \delta(x_0,y)^{\alpha-1}| \cdot d\mu(y)
\]

\[
+ \int_{\delta(x,y) \geq \epsilon \text{ and } \delta(x_0,y) < \epsilon} \frac{b(y) - b_\epsilon(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y)
\]

\[
- \int_{\delta(x_0,y) \geq \epsilon \text{ and } \delta(x,y) < \epsilon} \frac{b(y) - b_\epsilon(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y) d\mu(y)
\]

(2.28)

Note that, since \( \sigma > 2k^2(1+k(2k+1)) \), \( \delta(x_0,y) \leq 2k\delta(x_0,x) \) provided that \( \delta(x_0,y) > \epsilon \). Hence, according to [11, Lemma II.3] and Lemma 2.4, since \( \delta(x_0,x_1) < (1/2k)\epsilon \), we have,

\[
\int_{\delta(x,y) \geq \epsilon \text{ and } \delta(x_0,y) \geq \epsilon} |b(y) - b_\epsilon(x_0)| \cdot |f_2(y)| \cdot |\delta(x,y)^{\alpha-1} - \delta(x_0,y)^{\alpha-1}| \cdot d\mu(y)
\]

\[
\leq C \delta(x,x_0)^\rho \int_{X \setminus B(x_0,\epsilon)} |b(y) - b_\epsilon(x_0)| \cdot |f_2(y)| \cdot |\delta(x_0,y)^{\alpha-\rho-1}| d\mu(y)
\]

\[
\leq C \epsilon^\rho \int_{X \setminus B(x_0,\epsilon)} |b(y) - b_\epsilon(x_0)| \cdot |f_2(y)| \cdot |\delta(x_0,y)^{\alpha-\rho-1}| d\mu(y)
\]

\[
\leq C \|b\|_{*,p'} (M_p\alpha(|f|^p)(x_1))^{1/p} \leq Cy\lambda.
\]  

(2.29)

Moreover, \( \delta(x,y) < \epsilon \) implies that \( \delta(x_0,y) \leq \epsilon(k+1/2) \) and this inequality implies that \( \delta(x_1,y) \leq \epsilon(k(k+1/2)) + (1/2) \). Then, by taking into account the normality of
\((X, \delta, \mu)\), Holder inequality leads to
\[
\begin{align*}
\left| \int_{\delta(x,y) < \epsilon} & \frac{b(y) - b_\epsilon(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y) \\
- \int_{\delta(x_0,y) < \epsilon} & \frac{b(y) - b_\epsilon(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y) d\mu(y) \right| \\
\leq C \epsilon^{\alpha-1} & \int_{B(x_0, \epsilon(k+1/2))} |b(y) - b_\epsilon(x_0)| \|f_2(y)\| |d\mu(y)\|
\end{align*}
\]
(2.30)

\[
\leq C \frac{1}{\mu(B(x_0, \epsilon(k+1/2)))} \int_{B(x_0, \epsilon(k+1/2))} |b(y) - b_\epsilon(x_0)| \|f_2(y)\| |d\mu(y)\|
\leq \|b\|_{\ast, p'} (M_{p\alpha}(|f|^p)(x_1))^{1/p} \leq Cy\lambda.
\]

Finally, since \(x_0 \notin W_\lambda\), we have
\[
|I_3| \leq \lambda.
\] (2.31)

By combining (2.21), (2.27), and (2.31) we conclude that
\[
\sup_{\epsilon > d\sigma} \left| \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y) \right| \leq Cy\lambda + \lambda.
\] (2.32)

Assume now \(0 < \epsilon \leq d\sigma\). Let \(x \in B_j\). It is not hard to see that if \(y\) is in the support of \(f_2\) then \(\delta(x,y) \geq ((\alpha_1 - k)/k) d\) and \(\delta(x_0,y) \geq ((\alpha_1 - k^2(2k+1))/k) d\). We choose \(\omega > 0\) such that \(\omega < (\alpha_1 - k)/k\).

We can write
\[
\begin{align*}
&\int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y) = J_1 + J_2 + J_3,
\end{align*}
\]
(2.33)

where
\[
J_1 = \int_{X \setminus B(x, \epsilon)} \frac{b_{ood}(x_0) - b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y),
\]
\[
J_2 = \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_{ood}(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y) - \int_{X \setminus B(x_0, \epsilon)} \frac{b(y) - b_{ood}(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y) d\mu(y),
\]
\[
J_3 = \int_{X \setminus B(x_0, \epsilon)} \frac{b(y) - b_{ood}(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y) d\mu(y).
\] (2.34)

We will estimate \(J_i, i = 1, 2, 3\).

By proceeding as in the study of \(I_1\), since \(k(\sigma + 1) < \alpha_2\), we obtain
\[
\begin{align*}
|J_1| &\leq C |b_{ood}(x_0) - b_\epsilon(x)| I_\alpha(|f|)(x_1) \\
&\leq C \frac{1}{\mu(B(x, \epsilon))} \int_{B(x, \epsilon)} |b(y) - b_{ood}(x_0)| \chi_{B_j^2}(y) d\mu(y) I_\alpha(|f|)(x_1) \\
&\leq CM \left( (b - b_{ood}(x_0)) \chi_{B_j^2}(x) I_\alpha(|f|)(x_1) \right).
\] (2.35)
On the other hand, we have that

\[
J_2 = \int_{X \setminus B(x, \varepsilon)} \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x, y)^{1-\alpha}} f_2(y) d\mu(y) - \int_{X \setminus B(x, \varepsilon)} \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x, y)^{1-\alpha}} f_2(y) d\mu(y) \\
= \int_{\delta(x, y) \geq \varepsilon \text{ and } \delta(x_0, y) \geq \varepsilon} (b(y) - b_{\text{wod}}(x_0)) f_2(y) \left( \delta(x, y)^{\alpha-1} - \delta(x, y)^{\alpha-1} \right) d\mu(y) \\
+ \int_{\delta(x, y) \geq \varepsilon \text{ and } \delta(x_0, y) < \varepsilon} \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x, y)^{1-\alpha}} f_2(y) d\mu(y) \\
- \int_{\delta(x, y) < \varepsilon \text{ and } \delta(x_0, y) \geq \varepsilon} \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x_0, y)^{1-\alpha}} f_2(y) d\mu(y).
\]

(2.36)

Since \((X, \delta, \mu)\) is a space of homogeneous type of order \(\rho \in (0, 1)\), by virtue of [11, Lemma 2.3], we have

\[
\left| \int_{\delta(x, y) \geq \varepsilon \text{ and } \delta(x_0, y) \geq \varepsilon} (b(y) - b_{\text{wod}}(x_0)) f_2(y) \left( \delta(x, y)^{\alpha-1} - \delta(x_0, y)^{\alpha-1} \right) d\mu(y) \right| \\
\leq C \delta(x, x_0)^\rho \int_{\delta(x, y) \geq \varepsilon \text{ and } \delta(x_0, y) \geq \varepsilon} |b(y) - b_{\text{wod}}(x_0)| |f_2(y)| \delta(x, y)^{\alpha-\rho-1} d\mu(y),
\]

(2.37)

because if \(y\) is in the support of \(f_2\), then \(\delta(x, y) \geq ((\alpha_1 - 1)/k) d \geq 2k^2(2k + 1) + 1) d \geq 2k \delta(x_0, x)\). Hence, since \(y \in \text{supp} f_2\) implies that \(\delta(x_1, y) > \omega d\), by proceeding as in the proof of Lemma 2.4 we conclude

\[
\left| \int_{\delta(x, y) \geq \varepsilon \text{ and } \delta(x_0, y) \geq \varepsilon} (b(y) - b_{\text{wod}}(x_0)) f_2(y) \left( \delta(x, y)^{\alpha-1} - \delta(x_0, y)^{\alpha-1} \right) d\mu(y) \right| \\
\leq C \delta(x, x_0)^\rho \int_{\delta(x_1, y) > \omega d} \frac{|b(y) - b_{\text{wod}}(x_0)|}{\delta(x_1, y)^{1+\rho-\alpha}} |f(y)| d\mu(y) \\
\leq C \|b\|_{s, \rho'} (M_{\rho'\alpha}(|f|^p(x_1)))^{1/p} \leq C \gamma_\Lambda.
\]

(2.38)

Also, since if \(\delta(x, y) < \varepsilon\), then \(\delta(x_0, y) < dk(k(2k + 1) + 1) + \sigma\) and since \(\omega < \alpha_1\), we have

\[
\left| \int_{\delta(x, y) \leq \varepsilon \text{ and } \delta(x_0, y) < \varepsilon} \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x, y)^{1-\alpha}} f_2(y) d\mu(y) \\
- \int_{\delta(x, y) < \varepsilon \text{ and } \delta(x_0, y) \geq \varepsilon} \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x_0, y)^{1-\alpha}} f_2(y) d\mu(y) \right| \\
\leq \int_{\omega d < \delta(x_0, y) < k(k(2k + 1) + 1) + \sigma} |b(y) - b_{\text{wod}}(x_0)| |f_2(y)| \\
\times (\delta(x, y)^{\alpha-1} + \delta(x_0, y)^{\alpha-1}) d\mu(y) \\
\leq C \int_{\omega d < \delta(x_0, y) < k(k(2k + 1) + 1) + \sigma} \frac{|b(y) - b_{\text{wod}}(x_0)|}{\delta(x_0, y)^{1-\alpha}} |f_2(y)| d\mu(y).
\]

(2.39)
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Now by proceeding as in the proof of Lemma 2.4 we obtain that
\[
\left| \int_{\delta(x,y) \geq \epsilon} b(y) - b_{\text{mod}}(x_0) \frac{f_2(y)}{\delta(x,y)^{1-\alpha}} \, d\mu(y) \right| - \left| \int_{\delta(x,y) < \epsilon} b(y) - b_{\text{mod}}(x_0) \frac{f_2(y)}{\delta(x_0,y)^{1-\alpha}} \, d\mu(y) \right| \\
\leq C \|b\|_{*,p'} \left( M_{p\alpha}(\|f\|_p) (x_1) \right)^{1/p} \leq C \gamma \lambda. \tag{2.40}
\]

In a similar way we can see that
\[
\left| J_3 \right| \leq \int_{X \setminus B(x_0,\omega_d)} \frac{|b(y) - b_{\text{mod}}(x_0)|}{\delta(x,y)^{1-\alpha}} |f_2(y)| \, d\mu(y) \\
\leq C \|b\|_{*,p'} \left( M_{p\alpha}(\|f\|_p) (x_1) \right)^{1/p} \leq C \gamma \lambda. \tag{2.41}
\]

By combining the above estimates we can conclude
\[
\sup_{0 < \epsilon < \alpha \epsilon} \left| \int_{X \setminus B(x,\epsilon)} \frac{b(y) - b_{\text{mod}}(x)}{\delta(x,y)^{1-\alpha}} f_2(y) \, d\mu(y) \right| \\
\leq C \left( \lambda \gamma + I_{\alpha}(\|f\|_1) (x_1) M \left( (b - b_{\text{mod}}(x_0)) \chi_{B_j^c} \right)(x) \right). \tag{2.42}
\]

From (2.32) and (2.42) follows that for every \( x \in B_j \)
\[
C(b, f_2)(x) \leq C \left( \lambda \gamma + \lambda + I_{\alpha}(\|f\|_1) (x_1) M \left( (b - b_{\text{mod}}(x_0)) \chi_{B_j^c} \right)(x) \right). \tag{2.43}
\]

Hence if \( \beta \) is large enough, then according to Lemma 2.2 and since \( \mu \) is doubling
\[
\mu \left( \{ x \in B_j : C(b, f_2)(x) > \beta \lambda \} \right) \\
\leq \mu \left( \{ x \in B_j : I_{\alpha}(\|f\|_1)(x_1) M \left( (b - b_{\text{mod}}(x_0)) \chi_{B_j^c} \right)(x) > \lambda \} \right) \\
\leq C \lambda^{-1} I_{\alpha}(\|f\|_1)(x_1) \int_{B_j^c} |b(y) - b_{\text{mod}}(x_0)| \, d\mu(y) \\
\leq C \lambda^{-1} I_{\alpha}(\|f\|_1)(x_1) \|b\|_{*,p'} \mu(B_j) \leq C \gamma \mu(B_j). \tag{2.44}
\]

Thus we obtain that for \( \beta \geq \beta_0 \) and \( \gamma < 1 \), where \( \beta_0 \) is large enough,
\[
\mu(B_j \cap E_{\lambda}(\beta,y)) \leq C \gamma \mu(B_j). \tag{2.45}
\]

Hence
\[
\mu(B \cap E_{\lambda}(\beta,y)) \leq C \gamma \sum_{j=1}^{\infty} \mu(B_j) \leq C \gamma \mu(W_{\lambda}), \quad \beta \geq \beta_0, \, \gamma < 1. \tag{2.46}
\]

Arbitrariness of \( B \) allows to conclude that
\[
\mu(E_{\lambda}(\beta,y)) \leq C \gamma \mu(W_{\lambda}), \quad \beta \geq \beta_0, \, \gamma < 1, \tag{2.47}
\]
and the proof is finished. \( \square \)
PROOF OF THEOREM 1.1. To prove Theorem 1.1 we proceed as in the proof of [6, Theorem III]. We start proving that the operator $C(b, f)$ is bounded from $L^p(X, \mu)$ into $L^q(X, \mu)$, when $1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$. Assume that $b \in L^\infty(X, \mu)$.

Let $1 < p_1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$. Assume firstly that $\mu(X) = \infty$. According to Lemma 2.6, $f \in L^p(X, \mu)$ we have

$$\int_X (C(b, f)(x))^q d\mu(x)$$

provided that $\beta \geq \beta_0$ and $0 < \gamma < 1$, where $\beta_0$ is given in Lemma 2.6.

Hence by (2.4) and Lemma 2.1 and by taking $\gamma$ so small we can conclude that

$$\|C(b, f)\|_q \leq C \|b\|_{s, p'} \left( \|I_\alpha(\|f\|)(x)\| + \|M_{p_1, \alpha}(|f|^{p_1})(x)\|^{1/p_1}\right).$$

According to Lemmas 2.1 and 2.2 it follows

$$\|C(b, f)\|_q \leq C \|b\|_{s, p'} \|f\|_p.$$

Suppose now that $\mu(X) < \infty$. Since $C(b, f) = C(b - a, f)$, for every $a \in \mathbb{C}$, we can assume, without loss of generality, that $\int_X b \, d\mu = 0$. Then Lemma 2.6 leads, for every $f \in L^p(X, \mu)$, to

$$\int_X (C(b, f)(x))^q d\mu(x)$$

when $\beta \geq \beta_0$ and $0 < \gamma < 1$, $\beta_0$ being as in Lemma 2.6.

Thus we deduce from Lemmas 2.1 and 2.2 that

$$\|C(b, f)\|_q \leq C \|b\|_{s, p'} \|f\|_p.$$
Now we note that
\[
[b, I_\alpha] (f)(x) = \lim_{\epsilon \to 0^+} \left( b(x) \int_{X \setminus B(x,\epsilon)} \frac{f(y)}{\delta(x,y)^{1-\alpha}} d\mu(y) - \int_{X \setminus B(x,\epsilon)} \frac{b(y)f(y)}{\delta(x,y)^{1-\alpha}} d\mu(y) \right)
\]
\[
= \lim_{\epsilon \to 0^+} \left( (b(x) - b_\epsilon(x)) \int_{X \setminus B(x,\epsilon)} \frac{f(y)}{\delta(x,y)^{1-\alpha}} d\mu(y) \right)
\]
\[
- \int_{X \setminus B(x,\epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f(y) d\mu(y) \right)
\]
\[
= - \lim_{\epsilon \to 0^+} \int_{X \setminus B(x,\epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f(y) d\mu(y),
\]
for every \( f \in L^p(X,\mu) \), and a.e. \( x \in X \).

Then
\[
\| [b, I_\alpha] \|_q \leq \| C(b,f) \|_q,
\]
(2.54)

for each \( f \in L^p(X,\mu) \).

To finish the proof it is sufficient to take into account [3, Lemma 2.5] and Fatou's lemma.

\[\Box\]

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