GENERALIZATIONS OF p-VALENT FUNCTIONS
VIA THE HADAMARD PRODUCT

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(Received July 1, 1981)

ABSTRACT. The classes of univalent prestarlike functions $R_\alpha$, $\alpha \geq -1$, of Ruscheweyh [1] and a certain generalization of $R_\alpha$ were studied recently by Al-Amiri [2]. The author studies, among other things, the classes of p-valent functions $R(\alpha + p - 1)$ where $p$ is a positive integer and $\alpha$ is any integer with $\alpha + p > 0$. The author shows in particular that $R(\alpha + p) \subset R(\alpha + p - 1)$ and also obtains the radius of $R(\alpha + p)$ for the class $R(\alpha + p - 1)$.

KEY WORDS AND PHRASES. p-valent starlike functions, p-valent close-to-convex functions, Hadamard product.

AMS (MOS) SUBJECT CLASSIFICATION (1980) CODES. Primary 30C45.

1. INTRODUCTION.

The classes of univalent prestarlike functions $R_\alpha$, $\alpha \geq -1$, were studied by various authors [1,2]. The author extends these classes to the classes of p-valent starlike functions $R(\alpha + p - 1)$, where $p$ is a positive integer and $\alpha$ is any integer greater than $-p$. The present studies give, along with other results, a method to determine the radius of $R(\alpha + p)$ for the class $R(\alpha + p - 1)$.

Let $A_p$ denote the class of regular functions in the unit disc $D = \{z: |z| < 1\}$ having the power series

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \text{ a positive integer, } z \in D. \quad (1.1)$$
We denote by $S^*(\beta)$, the subclass of $A_1$ whose members are starlike of order $\beta$, $0 \leq \beta < 1$.

Ruscheweyh [1] introduced the following classes 'K' of univalent prestarlike functions:

$$ K_\alpha = \{ f(z) \mid f(z) \in A_1 \text{ and } \Re \left( \frac{zf(z)}{(z-z^*)^{\frac{\alpha+1}{\alpha}}} \right) > \frac{\alpha + 1}{2}, \quad z \in D \}, $$

$\alpha \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$; where $F^{(n)}$ denotes the $n$-th derivative of the function $F$. As observed by Ruscheweyh, $f \in K_\alpha$ if and only if $\Re \left( \frac{D^{\alpha+1}f(z)}{D^\alpha f(z)} \right) > \frac{1}{2}$, $z \in D$ where $D^\alpha f(z) = f(z)^* \frac{z}{(1-z)^{\alpha+1}}$. Here '*' denotes the Hadamard product of two regular functions, that is to say if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then $f(z)^* g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

Ruscheweyh proved that $K_{\alpha+1} \subset K_\alpha$ and $K_0 = S^*(\frac{1}{2})$. Hence for each $\alpha \in \mathbb{N}_0$, $K_\alpha$ is a subclass of $S^*(\frac{1}{2})$. Recently, Al-Amiri [2] studied a certain generalization of $K_\alpha$, in particular he obtained the radius of $K_{\alpha+1}$ in $K_\alpha$, $\alpha \in \mathbb{N}_0$. Further Singh and Singh [3] extended the classes $K_\alpha$ to the classes $R_\alpha$, where

$$ R_\alpha = \{ f(z) \mid f(z) \in A_1 \text{ and } \Re \left( \frac{D^{\alpha+1}f(z)}{D^\alpha f(z)} \right) > \frac{\alpha}{\alpha + 1}, \quad z \in D \}, \quad \alpha \in \mathbb{N}_0. $$

They observed that $R_\alpha$ is a subclass of $S^*(0)$. In this note, we extend their ideas to the class of $p$-valent functions.

We call a function $f(z) \in A_p$ to be $p$-valent starlike if it satisfies

$$ \Re \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in D. $$

Further, we say that a function $f(z) \in A_p$ is $p$-valent close-to-convex if there exists a $p$-valent starlike function $g(z)$ for which

$$ \Re \left( \frac{zf'(z)}{g(z)} \right) > 0, \quad z \in D. $$

Let $R(\alpha + p - 1)$ denote the class of functions $f(z) \in A_p$ satisfying

$$ \Re \left[ \left( \frac{zf(z)}{(z-z^*)^{\alpha+p-1}} \right)^{\frac{(\alpha+p)(\alpha+p-1)}{\alpha+p-1}} \right] > \alpha + p - 1, \quad z \in D, \quad (1.2) $$

where $\alpha$ is any integer greater that $-p$. In Section 2 we shall show that

$$ R(\alpha + p) \subset R(\alpha + p - 1). \quad (1.3) $$

Since $R(0)$ is the class of functions which satisfy

$$ \Re \left( \frac{zf'(z)}{f(z)} \right) > p - 1 \geq 0, $$
it follows by our definition taken from [4] that such functions are \( p \)-valent star-like. Hence (1.3) implies that \( R(\alpha + p - 1) \) contains \( p \)-valent starlike functions.

We denote by \( H(\alpha + p - 1) \), the classes of functions \( f(z) \in A_p \) that satisfy the condition

\[
\Re \left[ \frac{(z^\alpha f(z))^{(\alpha + p)} - \alpha(z^{\alpha - 1} f(z))^{(\alpha + p - 1)}}{(z^{\alpha - 1} g(z))^{(\alpha + p - 1)}} \right] > \frac{\alpha + p - 1}{\alpha + p}, \quad z \in D, \quad (1.4)
\]

for some \( g(z) \in R(\alpha + p - 1) \), \( \alpha \) integer greater than \(-p\).

In Section 4 we shall show that

\[
H(\alpha + p) \subseteq H(\alpha + p - 1). \quad (1.5)
\]

Again since \( H(0) \) is the class of functions \( f \) that satisfy \( \Re \frac{z f'(z)}{g(z)} > 0 \), where \( g \) is starlike, (1.5) implies that \( H(\alpha + p - 1) \) contains \( p \)-valent close-to-convex functions.

For \( f \in A_p \), define

\[
D^{\alpha + p - 1} f(z) = f(z)^* \frac{z^p}{(1 - z)^{\alpha + p - 1}}, \quad (1.6)
\]

where \( \alpha \) is any integer greater than \(-p\). Then

\[
D^{\alpha + p - 1} f(z) = z^p (\alpha f(z))^{\alpha + p - 1} / (\alpha + p - 1)! . \quad (1.7)
\]

It can be shown that (1.6) yields the following identity

\[
z(D^{\alpha + p - 1} f(z))' = (\alpha + p) D^{\alpha + p} f(z) - \alpha (D^{\alpha + p - 1} f(z)). \quad (1.8)
\]

From (1.2) and (1.7) it follows that a function \( f \) in \( A_p \) belongs to \( R(\alpha + p - 1) \) if and only if

\[
\Re \frac{D^{\alpha + p} f(z)}{D^{\alpha + p - 1} f(z)} > \frac{\alpha + p - 1}{\alpha + p}. \quad (1.9)
\]

Note that for \( p = 1 \), the classes \( R(\alpha + p - 1) \) reduce to the classes \( R_\alpha \) of Singh and Singh [3]. Hence our results are generalizations of Singh and Singh.

From (1.4) and (1.7), it follows that a function \( f \) in \( A_p \) belongs to \( H(\alpha + p - 1) \) if and only if

\[
\Re \left[ \frac{z(D^{\alpha + p - 1} f(z))'}{D^{\alpha + p - 1} g(z)} \right] > \frac{\alpha + p - 1}{\alpha + p}, \quad (1.10)
\]

for some \( g \in R(\alpha + p - 1) \).
In Sections 3 and 4 we shall describe some special elements of $R(\alpha + p - 1)$ and $H(\alpha + p - 1)$, respectively. These elements have integral representations. In Section 5, we introduce the classes $R_{\frac{1}{2}}(\alpha + p - 1)$ via the Hadamard product. Also the radii of $R(\alpha + p)$ in $R(\alpha + p - 1)$ and of $R_{\frac{1}{2}}(\alpha + p)$ in $R_{\frac{1}{2}}(\alpha + p - 1)$ are determined. In Section 6, the classes $R_{\frac{1}{2}}(\alpha + p - 1, \beta)$ which are extensions of the classes $R_{\frac{1}{2}}(\alpha + p - 1)$, are given. Many authors have considered a variation of these classes, notably Ruscheweyh [1], Suffridge [5], Goel and Sohi [6]. However, this note basically uses the techniques given by Al-Amiri [2].

2. THE CLASSES $R(\alpha + p - 1)$.

We shall prove the following:

THEOREM 1. $R(\alpha + p) \subset R(\alpha + p - 1)$.

PROOF. Let $f \in R(\alpha + p)$. Define $w(z)$ by

$$\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = \frac{\alpha + p - 1}{\alpha + p} + \frac{1}{\alpha + p} \frac{1 - w(z)}{1 + w(z)}.$$  \hspace{1cm} (2.1)

Here $w(z)$ is a regular function in $D$ with $w(0) = 0$, $w(z) \neq -1$ for $z \in D$. It suffices to show that $|w(z)| < 1$, $z \in D$, since then (2.1) would imply that $f \in R(\alpha + p - 1)$.

Taking logarithmic derivative of both sides of (2.1) and using the identity (1.8) the following is obtained.

$$\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)} = \frac{1}{(\alpha+p+1)} \left[ 1 + \frac{\alpha+p+(\alpha+p-2)w(z)}{1+w(z)} \right] - \frac{2zw'(z)}{(1+w(z))(\alpha+p+(\alpha+p-2)w(z))}. \hspace{1cm} (2.2)$$

The above equation must yield $|w(z)| < 1$ for all $z \in D$, for otherwise by using a lemma of Jack [7] one can obtain $z_0 \in D$ such that $z_0w'(z_0) = Kw(z_0)$, $|w(z_0)| = 1$ and $K \geq 1$. Consequently (2.2) would yield

$$\frac{D^{\alpha+p+1}f(z_0)}{D^{\alpha+p}f(z_0)} = \frac{1}{(\alpha+p+1)} + \frac{(\alpha+p)+(\alpha+p-2)w(z_0)}{(\alpha+p+1)(1+w(z_0))} - \frac{2Kw(z_0)}{(\alpha+p+1)(1+w(z_0))}.$$

$$\frac{(\alpha+p+(\alpha+p-2)w(z_0))}{|\alpha+p+(\alpha+p-2)w(z_0)|^2}.$$
Since
\[ \text{Re} \left( \frac{1}{1 + w(z_0)} \right) = \frac{1}{2}, \quad \text{Re} \left( \frac{w(z_0)}{1 + w(z_0)} \right) = \frac{1}{2}, \]
the above equation implies
\[ \frac{D^{\alpha+p+1}f(z_0)}{D^{\alpha+p}f(z_0)} \leq \frac{\alpha + p}{\alpha + p + 1}. \]
This is a contradiction to the assumption that \( f \in R(\alpha + p) \). Hence \( f \in R(\alpha + p - 1) \).
This completes the proof of Theorem 1.

3. SPECIAL ELEMENTS OF \( R(\alpha + p - 1) \).

In this section we form special elements of the classes \( R(\alpha + p - 1) \) via the Hadamard product of elements of \( R(\alpha + p - 1) \) and \( h_\gamma(z) \), where
\[ h_\gamma(z) = \sum_{j=p}^{\infty} \gamma + \frac{p}{j} \cdot z^j, \quad \text{Re} \gamma > -p. \]

**THEOREM 2.** Let \( f \in A_p \) satisfy the condition
\[ \frac{\text{Re} D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} > \frac{2((\gamma+p-1)(\gamma+p-1)-(\text{Re}D+P)f(z))}{2(\alpha+p)(\gamma+p-1)}, \quad z \in D, \quad (3.1) \]
\( p \) a positive integer, \( \alpha \) any integer greater than \(-p\) and \( \gamma \geq -p + 2 \).

Then
\[ F(z) = f(z)*h_\gamma(z) = \gamma + \frac{p}{z^\gamma} \cdot \int_0^z t^{\gamma-1} f(t) dt \quad (3.2) \]
belongs to \( R(\alpha + p - 1) \).

**PROOF.** Let \( f \in A_p \) satisfy the condition (3.1). From (3.2) we obtain
\[ z(D^{\alpha+p}F(z))' + \gamma D^{\gamma+p}F(z) = (p+\gamma)D^{\alpha+p}f(z), \quad (3.3) \]
and
\[ z(D^{\alpha+p-1}F(z))' + \gamma D^{\gamma+p-1}F(z) = (p+\gamma)D^{\alpha+p-1}f(z). \quad (3.4) \]
Define \( w(z) \) by
\[ \frac{D^{\alpha+p}F(z)}{D^{\alpha+p-1}F(z)} = \frac{\alpha + p - 1}{\alpha + p} + \frac{1}{\alpha + p} \cdot \frac{1 - w(z)}{1 + w(z)}. \quad (3.5) \]
Here \( w(z) \) is a regular function in \( D \) with \( w(0) = 0 \), \( w(z) \neq -1 \) for \( z \in D \). It suffices to show that \( |w(z)| < 1 \), \( z \in D \).

Taking the logarithmic derivative of (3.5) and using (1.8) for \( F(z) \) one can get
Now (3.3) and (3.6) yield
\[ (p+\gamma)D^{\alpha+p}f(z) = D^{\alpha+p}F(z) \cdot \left[ \gamma - \alpha + \frac{(\alpha+p)+(\alpha+p-2)w(z)}{1+w(z)} \right] - \frac{2zw'(z)}{(1+w(z))(\alpha+p+(\alpha+p-2)w(z))}. \] (3.7)

Use (3.4) and (1.8) to eliminate the derivative and then apply (3.5) to get
\[ (p+\gamma)D^{\alpha+p-1}f(z) = D^{\alpha+p-1}F(z) \cdot \left[ \gamma - \alpha + \frac{(\alpha+p)+(\alpha+p-2)w(z)}{1+w(z)} \right]. \] (3.8)

Therefore (3.7), (3.8) and (3.5) give
\[ \frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = \frac{\alpha + p - 1}{\alpha + p} + \frac{1}{1 + w(z)} - \frac{2zw'(z)}{(\alpha+p)(1+w(z))} \frac{(\gamma+p)+(\gamma+p-2)w(z)}{|\gamma+p+(\gamma+p-2)w(z)|^2}. \] (3.9)

Equation (3.9) should yield $|w(z)| < 1$ for all $z \in D$, for otherwise by Jack's lemma there exists $z_0 \in D$ with $z_0w'(z_0) \leq K w(z_0)$, $K \geq 1$, and $|w(z_0)| = 1$. Applying this to (3.9) it follows that
\[
\text{Re} \left[ \frac{D^{\alpha+p}f(z_0)}{D^{\alpha+p-1}f(z_0)} \right] \leq \frac{\alpha + p - 1}{\alpha + p} - \frac{2}{(\alpha+p)} \frac{\gamma + p - 1}{4(\gamma+p-1)^2} = \frac{2(\gamma+p-1)(\alpha+p-1) - 1}{2(\alpha+p)(\gamma+p-1)}.
\]

This contradicts the assumption on $f$ given by (3.1). Hence $F \in R(\alpha + p - 1)$. This completes the proof of Theorem 2.

REMARK 1. For $\gamma = 1$ and $p = 1$, Theorem 2 reduces to a result given in [3].

The following special cases of Theorem 2 represent some improvement on theorems due to Libera [8] in the sense that much weaker assumptions produce the same results.

By taking $\alpha = 0$, $p = 1$ in Theorem 2 we get

COROLLARY 1. Let $f \in A_1$ be such that $\text{Re} \left[ \frac{zf'(z)}{f(z)} \right] > -\frac{1}{2\gamma}$, $\gamma \geq 1$, $z \in D$. Then $F$ is starlike in $D$, where
For $\alpha = 1$, $p = 1$, Theorem 2 reduces to

**COROLLARY 2.** Let $f \in A_1$ be such that $\Re \left[ 1 + \frac{z f''(z)}{f'(z)} \right] > -\frac{1}{2\gamma}$, $\gamma \geq 1$, $z \in D$. Then $F(z)$ as given in (3.10) above is convex in $D$.

Using the technique employed in the proof of Theorem 1 and Corollary 2 we obtain the following result.

**COROLLARY 3.** Let $f \in A_1$ be such that $\Re \left[ 1 + \frac{z g''(z)}{g'(z)} \right] > -\frac{1}{2\gamma}$, $\gamma \geq 1$, $z \in D$. Then $F(z)$ as given by (3.10), is close-to-convex, i.e., $\Re \frac{F'(z)}{G'(z)} > 0$, $z \in D$ and where $G(z)$ is the convex function given by

$$G(z) = \frac{\gamma + 1}{z^\gamma} \cdot \int_0^z t^{\gamma-1} g(t) dt.$$
This implies that
\[ \text{Re} \frac{D^{\alpha+p+1} F(z)}{D^{\alpha+p} F(z)} > \frac{\alpha + p}{\alpha + p + 1}, \quad z \in D. \]

Hence \( F(z) \in R(\alpha + p) \), and this completes the proof of Theorem 4.

REMARK 2. For \( p = 1 \), Theorem 4 reduces to a result of Singh and Singh [3].

4. THE CLASSES \( H(\alpha + p - 1) \).

We state without proof Theorems 5 and 6 since their proofs use the same technique employed in Theorem 1. See Section 1 for the definition of the classes \( H(\alpha + p - 1) \).

THEOREM 5. \( H(\alpha + p) \subset H(\alpha + p - 1) \).

THEOREM 6. If \( p \) is any positive integer, \( \alpha \) is any integer greater than \(-p\), and \( \text{Re} \gamma \geq -p + 1 \), then
\[ F(z) = f(z) \cdot h_{\gamma}(z) = \frac{p + \gamma}{z - \gamma} \cdot \int_0^z \frac{t^{\gamma-1} f(t) dt}{\gamma} \in H(\alpha + p - 1) \]
whenever \( f(z) \in H(\alpha + p - 1) \).

5. RADII OF THE CLASSES \( R(\alpha + p) \) AND \( R_2(\alpha + p) \).

Because discussing the problem concerning the radii of the classes \( R(\alpha + p) \) and \( R_2(\alpha + p) \) we define the classes \( R_2(\alpha + p - 1) \). \( R_2(\alpha + p - 1) \) contains functions \( f(z) \in A_p \) that satisfy the condition
\[
\text{Re} \left[ \frac{(z f(z))^{\alpha+p}}{(z^{-1} f(z))^{\alpha+p-1}} \right] > \frac{\alpha + p}{2}, \quad z \in D, \tag{5.1}
\]
where \( \alpha \) is any integer greater than \(-p\). These classes have been studied by Goel and Sohi [6].

From (1.7) and (5.1), it follows that a function \( f \) in \( A_p \) belongs to \( R_2(\alpha + p - 1) \) if and only if
\[
\text{Re} \frac{D^{\alpha+p} f(z)}{D^{\alpha+p-1} f(z)} > \frac{1}{2}. \tag{5.2}
\]

Our main interest is to determine the radius of the largest disc \( D(r) = \{ z : |z| < r \} \), \( 0 < r \leq 1 \) so that if \( f \in R(\alpha + p - 1) \) then
\[
\text{Re} \frac{D^{\alpha+p} f(z)}{D^{\alpha+p-1} f(z)} > \frac{\beta + p - 1}{\beta + p}, \quad \beta > \alpha, \quad z \in D(r). \]

A partial answer to this problem can be deduced by a simple appli-
cution of a lemma due to (Ruscheweyh and Singh) [9]:

**Lemma 1.** If $p(z)$ is an analytic function in the unit disc $D$ with $p(0) = 1$, $\Re p(z) > 0$ and also

$$|z| < \frac{|\mu + 1|}{A + (A^2 - |\mu|^2 - 1)^{1/2} |\mu|^2},$$

$$A = 2(S + 1)^2 + |\mu|^2 - 1.$$

Then we have

$$\Re \left[ \frac{p(z) + S \frac{zp'(z)}{p(z) + \mu}}{z} \right] > 0.$$

The bound given by (5.3) is best possible.

**Theorem 7.** Let $p$ be any positive integer, $\alpha$ any integer greater than $-p$. If $f(z) \in R(\alpha + p - 1)$ then

$$\Re \frac{D^{\alpha+p+1} f(z)}{D^{\alpha+p} f(z)} > \frac{\alpha + p}{\alpha + p + 1} \quad \text{for} \quad |z| < r_{\alpha,p},$$

where

$$r_{\alpha,p} = \frac{\alpha + p}{2 + \sqrt{3 + (\alpha+p-1)^2}}.$$

This result is sharp.

**Proof.** Let $f(z) \in R(\alpha + p - 1)$. We define the regular function $q(z)$ on $D$ by

$$D^{\alpha+p} f(z) = \frac{1}{(\alpha + p)} \left( q(z) + \alpha + p - 1 \right), \quad z \in D.$$

Therefore $q(0) = 1$ and $\Re q(z) > 0$ in $D$.

Taking logarithmic derivative of (5.5) and using (1.8) we get

$$\frac{D^{\alpha+p+1} f(z)}{D^{\alpha+p} f(z)} - \frac{\alpha + p}{\alpha + p + 1} = \frac{1}{(\alpha+p+1)} \left[ q(z) + \frac{zp'(z)}{q(z) + \alpha + p - 1} \right].$$

Using Lemma (1) with $S = 1$, $\mu = \alpha + p - 1$, (5.6) and (5.3) show that

$$\Re \left[ \frac{D^{\alpha+p+1} f(z)}{D^{\alpha+p} f(z)} \right] > \frac{\alpha + p}{\alpha + p + 1} \quad \text{for} \quad |z| < \frac{\alpha + p}{A + (A^2 - ((\alpha+p-1)^2 - 1)^{1/2})^{1/2}},$$

where

$$A = (\alpha + p)^2 - 2(\alpha + p) + 8.$$
Minor computations yield the following:

\[ A + (A^2 - ((\alpha + p - 1)^2 - 1)^2)^{1/2} = (2 + \sqrt{3 + (\alpha + p - 1)^2})^2. \tag{5.8} \]

Thus (5.7) yields the radius \( r_{\alpha, p} \) as given by (5.4).

The method of Al-Amiri [2] is used to determine the extremal functions. The extremal functions thus obtained for this theorem are rotations of \( f(z) \) where \( f(z) \) is given by

\[
D + P f(z) = \frac{1}{z} + \frac{1}{z} D f(z)
\]

This completes the proof of Theorem 7.

**REMARK 3.** For \( \alpha = 0, p = 1 \), Theorem 7 gives the well-known radius of convexity for the class of starlike functions: \( r_{0,1} = 2 - \sqrt{3} \).

Now an easy modification of the method used by Al-Amiri [2, Theorem 4] gives the following result.

**THEOREM 8.** Let \( p \) be any positive integer, \( \alpha \) any integer greater than \(-p\). If \( f(z) \in R_{\alpha, p}(\alpha + p - 1) \), then \( f(z) \) satisfies (5.2) with \( \alpha \) replaced by \( \alpha + 1 \) for \( |z| < r_{\alpha, p} \), where

\[
r_{\alpha, p} = \left( \frac{(\alpha + p - 1)^2 + 2(\alpha + p + 2)}{(\alpha + p + 3) + 2(\alpha + p + 2)^{1/2}} \right)^{1/2}
\]

This result is sharp.

**REMARK 4.** For \( p = 1 \), Theorem 8 becomes a special case of a result due to Al-Amiri [2, Theorem 4].

6. **THE CLASSES \( R_{\alpha, p}(\alpha + p - 1, \beta) \).**

By \( R_{\alpha, p}(\alpha + p - 1, \beta) \), we denote the classes of functions \( f(z) \in A_p \) that satisfy

\[
\text{Re} \left[ (1 - \beta) \frac{D^{\alpha + p} f(z)}{D^{\alpha + p - 1} f(z)} + \beta \frac{D^{\alpha + p + 1} f(z)}{D^{\alpha + p} f(z)} \right] > \frac{1}{2}, \quad z \in D, \tag{6.1}
\]

for some \( \beta \geq 0 \), \( p \) any positive integer and \( \alpha \) any integer greater than \(-p\). Again using the technique employed in [2], the following theorem is obtained.

**THEOREM 9.** Let \( p \) be any positive integer, \( \alpha \) any integer greater than \(-p\). If \( f(z) \in R_{\alpha, p}(\alpha + p - 1, \beta) \), then \( f(z) \) satisfies (6.1) for \( |z| < r_{\alpha, p, \beta} \) where
This result is sharp.

REMARK 5. For $\beta = 1$, Theorem 9 reduces to Theorem 8. Also for $p = 1$, Theorem 9 represents a special case of a theorem due to Al-Amiri [2, Theorem 8].

ACKNOWLEDGEMENTS. This paper forms a part of the author's doctoral thesis written at Bowling Green State University of Ohio at Bowling Green. The author would like to thank Professor Hassoon S. Al-Amiri for his guidance and direction.

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