ON WEAK CENTER GALOIS EXTENSIONS OF RINGS

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ABSTRACT. Let \( B \) be a ring with 1, \( C \) the center of \( B \), \( G \) a finite automorphism group of \( B \), and \( B^G \) the set of elements in \( B \) fixed under each element in \( G \). Then, the notion of a center Galois extension of \( B^G \) with Galois group \( G \) (i.e., \( C \) is a Galois algebra over \( C^G \) with Galois group \( G|C \cong G \)) is generalized to a weak center Galois extension with group \( G \), where \( B \) is called a weak center Galois extension with group \( G \) if \( BI_i = B \) for some idempotent in \( C \) and \( I_i = \{ c - g(c) | c \in C \} \) for each \( g_i \neq 1 \) in \( G \). It is shown that \( B \) is a weak center Galois extension with group \( G \) if and only if for each \( g_i \neq 1 \) in \( G \) there exists an idempotent \( e_i \) in \( C \) and \( \{ b_k e_i, c_k e_i \in C e_i, k = 1, 2, \ldots, m \} \) such that \( \sum_1^m b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i \) and \( g_i \) restricted to \( C(1 - e_i) \) is an identity, and a structure of a weak center Galois extension with group \( G \) is also given.

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1. Introduction. Galois theory for fields was generalized for rings in the sixties and seventies \([3, 4, 7, 8]\). Let \( B \) be a ring with 1, \( G = \{ g_1 = 1, g_2, \ldots, g_n \} \) an automorphism group of \( B \) of order \( n \) for some integer \( n \), \( C \) the center of \( B \), and \( B^G \) the set of elements in \( B \) fixed under each element in \( G \). There are several well-known classes of noncommutative Galois extensions: (1) the DeMeyer-Kanzaki Galois extension \( B \) (i.e., \( B \) is an Azumaya \( C \)-algebra which is a Galois algebra with Galois group \( G|C \cong G \)) \([3, 7]\), (2) the \( H \)-separable Galois extension \( B \) (i.e., \( B \) is a Galois and a \( H \)-separable extension of \( B^G \)) \([8]\), (3) the Azumaya Galois extension \( B \) (i.e., \( B \) is a Galois extension of \( B^G \) which is an Azumaya \( C^G \)-algebra) \([1]\), (4) the central Galois algebra \([3, 4, 7]\), and (5) the center Galois extension \( B \) (i.e., \( C \) is a Galois algebra over \( C^G \) with Galois group \( G|C \cong G \)) \([11]\). We note that a commutative Galois extension is a DeMeyer-Kanzaki Galois extension which is a center Galois extension. It is well known that \( C \) is a Galois extension of \( C^G \) if and only if the ideals generated by \( \{ c - g(c) | c \in C \} \) is \( C \) for each \( g \neq 1 \) in \( G \) \([2, Proposition 1.2, page 80]\). This fact was generalized in \([11]\) to a center Galois extension; that is, \( B \) is a center Galois extension of \( B^G \) if and only if the ideals of \( B \) generated by \( \{ c - g(c) | c \in C \} \) is \( B \), that is, \( BI_i = B, \) where \( I_i = \{ c - g_i(c) | c \in C \} \) for each \( g_i \neq 1 \) in \( G \) (for more about center Galois extensions, see \([5, 6, 9, 10, 11]\)). Generalizing the condition that \( BI_i = B = B1 \) to that \( BI_i = Be_i \) for some idempotent \( e_i \) in \( C \) for each \( g_i \neq 1 \) in \( G \), we obtain a broader class of rings \( B \) than the class of center Galois extensions. This class of rings is called weak center Galois extensions. The purpose of the present paper is to give a characterization and a structure of a weak center Galois extension \( B \) with group \( G \). We shall show that \( B \) is a weak center Galois extension with group \( G \) if and only if for each \( g_i \neq 1 \) in \( G \) there exists an idempotent \( e_i \) in \( C \) and \( \{ b_k e_i \in Be_i; c_k e_i \in C e_i, k = 1, 2, \ldots, m \} \) such that \( \sum_1^m b_k e_i g_i(c_k e_i) = \delta_{1,g_i} e_i \).
and $g_{1}$ restricted to $C(1-e_{i})$ is an identity. Next, we call $B$ a $T$-Galois extension of $B^{T}$ if there exist elements $\{a_{i}, b_{i}\}$ in $B$, $i = 1, 2, \ldots, m$ for some integer $m$ such that $\sum_{i=1}^{m} a_{i} g_{1}(b_{i}) = \delta_{1,g}$ for $g \in T \cup \{1\}$. We note that $T$ is not necessarily a subgroup of $G$. Let $B$ be a weak center Galois extension with group $G$. Then, we show that there exists a partition of $G - \{1\}$, $\{T_{j} \subset G, j = 1, 2, \ldots, h \}$ for some integer $h$ and some idempotents $e_{j} \in C$, $j = 1, 2, \ldots, h$ such that $B_{e_{j}}$ is a $T_{j}$-Galois extension of $(Be_{j})^{T_{j}}$. So $B = \sum_{j=1}^{h} B_{e_{j}} \oplus B(1 - \bigvee_{j=1}^{h} e_{j})$ such that $B_{e_{j}}$ is a $T_{j}$-Galois extension of $(Be_{j})^{T_{j}}$ for $j = 1, 2, \ldots, h$, where $\bigvee$ is the sum of the Boolean algebra of the idempotents in $C$. Moreover, when $G$ is abelian, $e_{j}$ can be taken as orthogonal idempotents in $C$ so that $\sum_{j=1}^{h} B_{e_{j}}$ is a direct sum. Furthermore, a sufficient condition is given for the existence of a subgroup $H_{j} \subset T_{j} \cup \{1\}$ for $j = 1, 2, \ldots, h$. In this case, $B_{e_{j}}$ is a $H_{j}$-Galois extension of $(Be_{j})^{T_{j}}$ with Galois group $H_{j}$.

2. Definitions and notation. Throughout this paper, $B$ represents a ring with 1, $G = \{g_{1}, g_{2}, \ldots, g_{n}\}$ an automorphism group of $B$ of order $n$ for some integer $n$, $C$ the center of $B$, and $B^{G}$ the set of elements in $B$ fixed under each element in $G$. We denote $I_{i} = \{c - g_{i}(c) \mid c \in C\}$ and $BI_{i}$ the ideal of $B$ generated by $I_{i}$ for $g_{i} \in G$. $B$ is called a $G$-Galois extension of $B^{G}$ if there exist elements $\{a_{i}, b_{i}\}$ in $B$, $i = 1, 2, \ldots, m$ for some integer $m$ such that $\sum_{i=1}^{m} a_{i} g_{1}(b_{i}) = \delta_{1,g}$. Such a set $\{a_{i}, b_{i}\}$ is called a $G$-Galois system for $B$. $B$ is called a weak center Galois extension of $B^{G}$ with group $G$ if $BI_{i} = B_{e_{i}}$ for some idempotent in $C$ for each $g_{i} \neq 1$ in $G$. For a subset $T$ (not necessarily a subgroup) of $G$, $B$ is called a $T$-Galois extension of $B^{T}$ if there exist elements $\{a_{i}, b_{i}\}$ in $B$, $i = 1, 2, \ldots, m$ for some integer $m$ such that $\sum_{i=1}^{m} a_{i} g_{1}(b_{i}) = \delta_{1,g}$ for $g \in T \cup \{1\}$. Such a set $\{a_{i}, b_{i}\}$ is called a $T$-Galois system for $B$. For a $B$-module $M$, we denote $\text{Ann}_{B}(M) = \{b \in B \mid bm = 0 \text{ for all } m \in M\}$.

3. Weak center Galois extensions. In [11], the present authors showed that a center Galois extension $B$ is equivalent to each of the following statements: (i) $BI_{i} = B$ for each $g_{i} \neq 1$ in $G$ and (ii) $B$ is a Galois extension of $B^{G}$ with a Galois system $\{b_{i} \in B, c_{i} \in C, i = 1, 2, \ldots, m\}$ for some integer $m$. In this section, we generalize this characterization to a weak center Galois extension $B$ with group $G$. We begin with the following lemma.

**Lemma 3.1.** If $B$ is a weak center Galois extension with group $G$, then

(1) $g_{i}$ restricted to $Be_{i}$ is an automorphism of $Be_{i}$.
(2) $B_{e_{i}}$ is a $\{g_{i}\}$-Galois extension of $(Be_{i})^{[g_{i}]}$.

**Proof.** (1) For any $b = \sum_{k=1}^{m} b_{k}(c_{k} - g_{i}(c_{k})) \in BI_{i} = Be_{i}$, where $b_{k} \in B$ and $c_{k} \in C$, $k = 1, 2, \ldots, m$ for some integer $m$, we have $g_{i}(b) = g_{i}(\sum_{k=1}^{m} b_{k}(c_{k} - g_{i}(c_{k}))) = \sum_{k=1}^{m} g_{i}(b_{k})(c_{k} - g_{i}(c_{k})) \in BI_{i} = Be_{i}$. Hence, $g_{i}(Be_{i}) \subset Be_{i}$. Thus, $g_{i}$ restricted to $Be_{i}$ is an automorphism of $Be_{i}$ since $g_{i}$ is an automorphism of $B$.

(2) Since $BI_{i} = Be_{i}$, there exist $\{b_{k} \in B, c_{k} \in C, k = 1, 2, \ldots, m\}$ for some integer $m$ such that $\sum_{k=1}^{m} b_{k}(c_{k} - g_{i}(c_{k})) = e_{i}$. Therefore, $\sum_{k=1}^{m} b_{k}c_{k} = e_{i} + \sum_{k=1}^{m} b_{k}g_{i}(c_{k})$. Let $b_{m+1} = -\sum_{k=1}^{m} b_{k}g_{i}(c_{k})$ and $c_{m+1} = 1$. Then $\sum_{k=1}^{m+1} b_{k}c_{k} = e_{i}$ and $\sum_{k=1}^{m+1} b_{k}g_{i}(c_{k}) = 0$. Noting that $e_{i}$ is the identity of $Be_{i}$ and $g_{i}$ restricted to $Be_{i}$ is an automorphism
of \( Be_i \) we have \( g_i(e_i) = e_i \). Hence, \( \sum_{k=1}^{m+1} b_ke_i g_i(c_k e_i) = \delta_{1,0} e_i \), that is, \( \{ b_k e_i; c_k e_i, k = 1, 2, \ldots, m + 1 \} \) is a \( \{ g_i \} \)-Galois system for \( Be_i \).

The following is an equivalent condition for a weak center Galois extension with group \( G \).

**Theorem 3.2.** \( B \) is a weak center Galois extension with group \( G \) (i.e., \( Bl = Be_i \) for some idempotent \( e_i \) in \( C \) for each \( g_i \neq 1 \) in \( G \)) if and only if for each \( g_i \neq 1 \) in \( G \) there exists an idempotent \( e_i \) in \( C \) and \( \{ b_k e_i; c_k e_i \in C e_i, k = 1, 2, \ldots, m \} \) such that \( \sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_{1,0} e_i \) and \( g_i \) restricted to \( C(1 - e_i) \) is an identity.

**Proof.** (\( \Rightarrow \)) By Lemma 3.1(2), \( Bl = Be_i \) contains a \( \{ g_i \} \)-Galois system \( \{ b_k e_i \in Be_i; c_k e_i \in C e_i, k = 1, 2, \ldots, m \} \) such that \( \sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_{1,0} e_i \). Next, we show that \( g_i \) restricted to \( C(1 - e_i) \) is an identity. In fact, by Lemma 3.1(1), \( g_i(e_i) = e_i \). Hence, for any \( c \in C \), \( g_i(c(1 - e_i)) = g_i((1 - e_i)c) = (c - g_i(c))(1 - e_i) \). Since \( g_i(e_i) = e_i \), \( c - g_i(c) = ce_i - g_i(ce_i) = (c - g_i(c)) e_i \) for all \( c \in C \). This proves that \( g_i \) restricted to \( C(1 - e_i) \) is an identity.

(\( \Leftarrow \)) By hypothesis, for each \( g_i \neq 1 \) in \( G \) there exists an idempotent \( e_i \) in \( C \) and \( \{ b_k e_i \in Be_i; c_k e_i \in C e_i, k = 1, 2, \ldots, m \} \) such that \( \sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_{1,0} e_i \). Hence, \( e_i = \sum_{k=1}^{m} b_k e_i(c_k e_i - g_i(c_k e_i)) \in Bl \). Hence, \( Be_i \subseteq Bl \). But \( e_i \) is an idempotent, so \( Be_i = Be_i e_i \subseteq Bl e_i \subseteq Be_i \). Thus, \( Be_i = Bl e_i \). Since \( g_i \) restricted to \( C(1 - e_i) \) is an identity, \( g_i(c(1 - e_i)) = c(1 - e_i) \) for all \( c \in C \) (in particular, \( g_i(e_i) = e_i \)). Hence, \( c - g_i(c) = ce_i - g_i(ce_i) = (c - g_i(c)) e_i \) for all \( c \in C \). This implies that \( Be_i = Bl e_i = Bl \).

Recall that \( B \) is called a \( T \)-Galois extension of \( B^T \) for a subset \( T \) (not necessary a subgroup) of \( G \) if \( B \) contains a \( T \)-Galois system. Next, we give a structure of a weak center Galois extension with group \( G \).

**Lemma 3.3.** Assume \( B \) is a weak center Galois extension with group \( G \). Let \( T_j = \{ g_i \in G \mid Bl = Be_j \} \), i.e., \( e_i = e_j \). Then \( Be_j \) is a \( T_j \)-Galois extension of \( (Be_j)^1 \) for each \( j \neq 1 \).

**Proof.** By the proof of Lemma 3.1(2), for each \( g_i \in T_j \), there is a \( \{ g_i \} \)-Galois system \( \{ b_k^{(i)} e_j; c_k^{(i)} e_j, k = 1, 2, \ldots, m_i \} \) for \( Be_j \), where \( b_k^{(i)} \in B \) and \( c_k^{(i)} \in C \), \( k = 1, 2, \ldots, m_i \) for some integer \( m_i \). Denote the elements in \( T_j \) by \( \{ g_{i_1}, g_{i_2}, \ldots, g_{i_l} \} \) for some integer \( t \). Let

\[
\begin{align*}
\sum_{k_1=1}^{m_{i_1}} \cdots \sum_{k_t=1}^{m_{i_t}} b_{k_1, k_2, \ldots, k_t} c_{k_1, k_2, \ldots, k_t} e_j &= \sum_{k_1=1}^{m_{i_1}} \cdots \sum_{k_t=1}^{m_{i_t}} \left( b_{k_1}^{(i_1)} b_{k_2}^{(i_2)} \cdots b_{k_t}^{(i_t)} e_j \right) \left( c_{k_1}^{(i_1)} c_{k_2}^{(i_2)} \cdots c_{k_t}^{(i_t)} e_j \right) \\
&= \sum_{k_1=1}^{m_{i_1}} \left( b_{k_1}^{(i_1)} e_j \right) \left( c_{k_1}^{(i_1)} e_j \right) \sum_{k_2=1}^{m_{i_2}} \left( b_{k_2}^{(i_2)} e_j \right) \left( c_{k_2}^{(i_2)} e_j \right) \cdots \sum_{k_t=1}^{m_{i_t}} \left( b_{k_t}^{(i_t)} e_j \right) \left( c_{k_t}^{(i_t)} e_j \right) \\
&= e_j,
\end{align*}
\]

(3.1)
and, for each $g_i \in T_j$,

$$
\sum_{k_1=1}^{m_{i_1}} \sum_{k_2=1}^{m_{i_2}} \cdots \sum_{k_t=1}^{m_{i_t}} b_{k_1,k_2,\ldots,k_t} g_i(c_{k_1,k_2,\ldots,k_t})
= \sum_{k_1=1}^{m_{i_1}} \sum_{k_2=1}^{m_{i_2}} \cdots \sum_{k_t=1}^{m_{i_t}} \left( b_{k_1}^{(i_1)} b_{k_2}^{(i_2)} \cdots b_{k_t}^{(i_t)} e_j \right) g_i(c_{k_1}^{(i_1)} c_{k_2}^{(i_2)} \cdots c_{k_t}^{(i_t)} e_j)
\sum_{k_1=1}^{m_{i_1}} \left( b_{k_1}^{(i_1)} e_j \right) g_i(c_{k_1}^{(i_1)} e_j) \sum_{k_2=1}^{m_{i_2}} \left( b_{k_2}^{(i_2)} e_j \right) g_i(c_{k_2}^{(i_2)} e_j) \cdots \sum_{k_t=1}^{m_{i_t}} \left( b_{k_t}^{(i_t)} e_j \right) g_i(c_{k_t}^{(i_t)} e_j)
= 0.
$$

Thus, $\{b_{k_1,k_2,\ldots,k_t}; c_{k_1,k_2,\ldots,k_t}, k_t = 1, 2, \ldots, m_{i_t} \text{ and } l = 1, 2, \ldots, t\}$ is a $T_j$-Galois system for $Be_j$. This completes the proof. \hfill \Box

**Theorem 3.4.** If $B$ is a weak center Galois extension with group $G$, then there exists a partition $\{T_j \subseteq G, j = 1, 2, \ldots, m\}$ of $G - \{1\}$ and a finite set of central idempotents $\{e'_i \mid i = 1, 2, \ldots, m\}$ such that (1) $Be'_j$ is a $T_j$-Galois extension of $(Be'_j)^T_j$, (2) $B = \sum_{j=1}^m Be'_j \oplus B(1 - \vee_{j=1}^m e'_j)$, where $\vee_{j=1}^m e'_j$ is the sum of $e_1', e_2', \ldots, e_m'$ in the Boolean algebra of all idempotents in $C$, and (3) $G|_{C(1 - \vee_{j=1}^m e'_j)} = \{1\}$.

**Proof.** (1) Since $BI_i = Be_i$ for some idempotent $e_i$ in $C$ for each $g_i \neq 1$ in $G$, we have a partition of all distinct idempotents $\{e_i \mid g_i \neq 1\}$ in $G$. Let $E = \{e_j \mid j = 1, 2, \ldots, m\}$ be the set of all distinct idempotents in $\{e_i \mid g_i \neq 1\}$ and let $T_j = \{g_i \in G \mid BI_i = Be'_j\}$, i.e., $e_i = e_j$. Then $Be'_j$ is a $T_j$-Galois extension of $(Be'_j)^T_j$ for each $j = 1, 2, \ldots, m$ by Lemma 3.3. Moreover, since $E = \{e_j \mid j = 1, 2, \ldots, m\}$ is the set of all distinct idempotents in $\{e_i \mid BI_i = Be_i\}$ for $g_i \neq 1$ in $G$, it is easy to see that $T_i \cap T_j = \emptyset$, the empty set for $i \neq j$, and that $\cup_{j=1}^m T_j = G - \{1\}$, that is, $\{T_j \subseteq G, j = 1, 2, \ldots, m\}$ is a partition of $G - \{1\}$.

Part (2) is an immediate consequence of part (1), and Theorem 3.2 implies part (3).

We remark that the partition of $G - \{1\}$, $\{T_j \subseteq G, j = 1, 2, \ldots, m\}$ is determined by the set of all distinct idempotents in $\{e_i \mid BI_i = Be_i\}$ for $g_i \neq 1$ in $G$.

When $G$ is abelian, we obtain a stronger structure of a weak center Galois extension with group $G$.

**Lemma 3.5.** Assume that $B$ is a weak center Galois extension with group $G$. If $G$ is abelian, then $g_j(e_i) = e_i$ for all $i, j = 2, 3, \ldots, n$.

**Proof.** For any $c - g_i(c) \in I_i$, $g_j(c - g_i(c)) = g_j(c) - g_i(g_j(c)) \in I_i$. Hence, $g_j(BI_i) \subseteq BI_i$. Thus, $g_j$ restricted to $BI_i (= Be_i)$ is an automorphism of $Be_i$ since $g_j$ is an automorphism of $B$. Therefore, $g_j(e_i) = e_i$. \hfill \Box

**Theorem 3.6.** Assume that $B$ is a weak center Galois extension with group $G$. If $G$ is abelian, then there exist orthogonal idempotents $\{f_i \mid i = 1, 2, \ldots, p\}$ for some integer $p$ and some subset $T^{(i)}$ of $G$, $i = 1, 2, \ldots, p$ such that $B = \oplus_{i=1}^p Bf_i \oplus B(1 - \vee_{i=1}^p f_i)$, where $\vee_{i=1}^p f_i$ is the sum of $f_1, f_2, \ldots, f_p$ in the Boolean algebra of all idempotents in $C$ and $Bf_i$ is a $T^{(i)}$-Galois extension of $(Bf_i)^{T^{(i)}}$ for $i = 1, 2, \ldots, p$. 

ON WEAK CENTER GALOIS EXTENSIONS OF RINGS 493

Proof. By Theorem 3.4, there exists a set of distinct idempotents \( E = \{ e'_j \mid j = 1, 2, \ldots, m \} \) in \( C \) and a partition \( \{ T_j \mid j = 1, 2, \ldots, m \} \) of \( G - \{ 1 \} \) such that \( Be'_j \) is a \( T_j \)-Galois extension of \( (Be'_j)^Tj \) for \( j = 1, 2, \ldots, m \). Now, let \( S \) be the Boolean subalgebra generated by \( E \) with all nonzero minimal elements \( f_1, f_2, \ldots, f_p \) in \( S \). Then, it is easy to see that \( f_i f_j = 0 \) for \( i \neq j \), and so \( f_1, f_2, \ldots, f_p \) are orthogonal idempotents in \( C \). For each \( f_i, i = 1, 2, \ldots, p, f_i = e'_{j_1} e'_{j_2} \cdots e'_{j_p} \). By Theorem 3.4, \( Be'_j \) is a \( T_j \)-Galois extension of \( (Be'_j)^Tj \) for each \( l = 1, 2, \ldots, p_i \) with a \( T_j \)-Galois system \( \{ b_{l_1}^{(l)}, e'_{j_1} \mid b_{l_1}^{(l)} \subseteq B, e'_{j_1} \subseteq C \} \), and \( t_1 = 1, 2, \ldots, m_1 \). Hence, by using the same patching method as given in Lemma 3.3, \( \{ b_{l_1l_2, \ldots, l_p} = b_{l_1}^{(l_1)} b_{l_2}^{(l_2)} \cdots b_{l_p}^{(l_p)} f_i; \ c_{l_1l_2, \ldots, l_p} = c_{l_1}^{(1)} c_{l_2}^{(2)} \cdots c_{l_p}^{(p)} f_i \mid t = 1, 2, \ldots, m_l \} \) is a \( T^{(l)} \)-Galois system for \( Bf_i \), where \( T^{(l)} = \cup_{l=1}^{k_i} T_j \). Thus, \( B = \oplus_{l=1}^{k_i} Bf_i \oplus B(1 - \cup_{l=1}^{k_i} f_i) \) such that \( Bf_i \) is a \( (T^{(l)}) \)-Galois extension of \( (Bf_i)^{T^{(l)}} \) for \( i = 1, 2, \ldots, p \) and \( \{ f_1, f_2, \ldots, f_p \} \) is a set of orthogonal idempotents in \( C \).

4. Special cases. We note that the \( T_i \)'s in Theorem 3.4 and \( T^{(i)} \)'s in Theorem 3.6 may not be subgroups of \( G \). Next, we give a sufficient condition for each \( T_i \cup \{ 1 \} \subset G \) containing a subgroup \( H_i \) so that \( Be_i \) is a \( H_i \)-Galois extension of \( (Be_i)^{H_i} \) with Galois group \( H_i \). Consequently, \( Be_i \) becomes a center Galois extension of \( (Be_i)^{H_i} \) with Galois group \( H_i \), and \( B \) is a center Galois extension of \( G \) with Galois group \( G \) if \( e_i = 1 \) for all \( g_i \). We first show a relation between \( B(1 - e_p), B(1 - e_q), \) and \( B(1 - e_i) \), where \( g_p g_q = g_i \) in \( G \).

Lemma 4.1. Let \( J_i = \{ b \in B \mid bc = g_i(c)b \text{ for all } c \in C \} \) for each \( g_i \in G \). Then, \( J_p J_q \subset J_i \) if \( g_p g_q = g_i \in G \).

Proof. Let \( a \in J_p \) and \( b \in J_q \). Then, for any \( c \in C, (ab)c = ag_q(c)b = g_p(g_q(c))(ab) = g_i(c)(ab) \), where \( g_p g_q = g_i \). Hence, \( ab \in J_i \). Thus, \( J_p J_q \subset J_i \).}

Corollary 4.2. If \( B \) is a weak center Galois extension with group \( G \), then \( B(1 - e_p)B(1 - e_q) \subset B(1 - e_i) \), where \( g_p g_q = g_i \in G \).

Proof. Since \( B \) is a weak center Galois extension with group \( G \), \( BI_i = Be_i \) for some idempotent \( e_i \) in \( C \) for each \( g_i \) in \( G \). But \( I_i = \{ c - g_i(c) \mid c \in C \} \), so \( J_i = \{ b \in B \mid bc = g_i(c)b \text{ for all } c \in C \} = \{ b \in B \mid bc = g_i(c) \} = 0 \text{ for all } c \in C \). Hence, \( J_i = \text{Ann}_b(I_i) = \text{Ann}_b(BI_i) = \text{Ann}_b(Be_i) = B(1 - e_i) \). Thus, by Lemma 4.1, we have \( B(1 - e_p)B(1 - e_q) \subset B(1 - e_i) \), where \( g_p g_q = g_i \in G \).

Theorem 4.3. Assume that \( B \) is a weak center Galois extension with group \( G \). Let \( T_i, \) for each \( i = 1, 2, \ldots, n, \) be the subset of \( G \) as given in Theorem 3.4 such that \( Be_i \) is a \( T_i \)-Galois extension of \( (Be_i)^{T_i} \), the Boolean subalgebra generated by \( \{ e_i \mid g_i \notin G \} \) with all nonzero minimal elements \( \{ f_1, f_2, \ldots, f_k \} \) in \( S \), and \( H_j = \{ 1 \} \cup \{ g_i \in G \mid e_i f_1 = f_j \text{ and } e_i f_1 = 0 \text{ for all } l \neq j \} \). Then, \( H_j \) is a subgroup of \( G \) for each \( j = 1, 2, \ldots, k \) such that \( g_i(f_i) = f_j \) for each \( g_i \in H_j \).

Proof. (1) For any \( g_p \) and \( g_q \) in \( H_j \), let \( g_p g_q = g_i \) for some \( g_i \in G \). We claim that \( g_i \in H_j \) if \( g_i \neq 1 \). Since \( g_i \neq 1, BI_i = Be_i \) for some idempotent \( e_i \neq 0 \) in \( C \). By Corollary 4.2, \( B(1 - e_p)B(1 - e_q) \subset B(1 - e_i) \). Therefore, in the Boolean algebra of all
idempotents in $C$ with operations $\wedge, \lor$, complement, and the relation $\prec, (1 - e_p)(1 - e_q) < (1 - e_t)$. So $e_t < e_p \lor e_q = e_p + e_q - e_p e_q$. Thus, $e_t = e_t(e_p + e_q - e_p e_q)$. Since $g_p, g_q \in H_j, e_p f_i = 0$ and $e_q f_i = 0$ for all $l \neq j$. Hence, $e_t f_i = e_t(e_p + e_q - e_p e_q) f_i = 0$ for all $l \neq j$. Moreover, since $S$ is the Boolean subalgebra generated by $\{e_t | g_i \neq 1 \in G\}$, there is at least one non-zero minimal element in $S$ less than $e_t$. But $e_t f_i = 0$ for all $l \neq j$, so $f_j$ must be less than $e_t$. Hence, $e_t f_j = f_j$. Thus, $g_1(= g_p g_q) \in H_j$, and so $H_j$ is a subgroup of $G$. Moreover, suppose $g_i \in H_j$. Then $e_i f_j = f_j$ and $e_i f_i = 0$ for all $l \neq j$. Hence, $g_i$ is greater than $f_j$, but not greater than $f_1$ for all $l \neq j$. Since $g_i(e_i) = e_i, g_i(f_j)$ is a non-zero minimal element in $S$ less than $e_t$. Thus, $g_i(f_j) = f_j$. □

**Corollary 4.4.** Keeping the notation in Theorem 4.3, if $H_j \neq \{1\}$ for $j = 1, 2, \ldots, p$, then $B = \oplus \sum_{j=1}^{p} B f_j \oplus B (1 - \lor_{j=1}^{p} f_j)$, where $\lor_{j=1}^{p} f_j$ is the sum of $f_1, f_2, \ldots, f_p$ in the Boolean algebra of all idempotents in $C$, such that $B f_j$ is a $H_j$-Galois extension of $(B f_j)^{H_j}$ with Galois group $H_j$ for $j = 1, 2, \ldots, p$.

**Corollary 4.5.** If $B f_j = B$ for each $g_j \neq 1$ in $G$, then $B$ is a center Galois extension of $B^G$ with Galois group $G$.

**Proof.** Since $e_2 = e_3 = \cdots = e_n, T_2 = T_3 = \cdots = T_n = G - \{1\}$, so $T_j \cup \{1\} = G$. Thus, $B$ is a Galois extension of $B^G$ with a Galois system $\{b_i \in B; c_i \in C, i = 1, 2, \ldots, m\}$ for some integer $m$, that is, $B$ is a center Galois extension of $B^G$ with Galois group $G$. □

If the order of each non-identity element in $G$ has order 2 (hence, $G$ is abelian), the following theorem shows that $T_i \cup \{1\}$ contains a subgroup of $G$ for each $g_j \neq 1$ in $T_i$.

**Theorem 4.6.** Assume that $B$ is a weak center Galois extension with group $G$. If each non-identity element $g_i$ in $G$ has order 2, then $T_i$ contains a subgroup of $H_i$ of order 2 for each $g_j \neq 1$ in $G$ such that $B e_i$ is a $H_i$-Galois extension of $(B e_i)^{H_i}$ with Galois group $H_i$.

**Proof.** Let $B f_i = B e_i$ for $g_i \neq 1$ in $G$. Then $H_i = \{1, g_i\}$ is a subgroup contained in $T_i \cup \{1\}$, where $T_i = \{g_k \in G | B f_k = B e_i\}$ as defined in Theorem 3.4. Since $B e_i$ is a $T_i$-Galois extension of $(B e_i)^{T_i}, B e_i$ is a $H_i$-Galois extension of $(B e_i)^{H_i}$ with Galois group $H_i$.

**Theorem 3.4** shows that a weak center Galois extension is a sum of $T_i$-Galois extensions for some $T_i \subset G$ and **Theorem 4.6** states a weak center Galois extension as a direct sum of center Galois extensions. The following is an example of a weak center Galois extension with group $G$ as given in **Theorem 4.6**, but not a Galois extension.

**Example 4.7.** Let $\mathbb{Q}$ be the rational field, $B = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$, and $G = \{g_1 = 1, g_2, g_3, g_4 = g_2 g_3\}$ such that $g_2(a_1, a_2, a_3, a_4, a_5) = (a_2, a_1, a_3, a_4, a_5)$ and $g_3(a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_4, a_3, a_5)$ for all $(a_1, a_2, a_3, a_4, a_5) \in B$. Then,

1. $B f_i = B e_i$ for each $g_i \neq 1$ in $G$, where $e_2 = (1, 1, 0, 0, 0), e_3 = (0, 0, 1, 1, 0)$, and $e_4 = (1, 1, 1, 1, 0)$. Hence, $B$ is a weak center Galois extension with group $G$.

2. $B$ is not a Galois extension since $G$ restricted to $\{0, 0, 0, 0, a\}$ is $\{a \in \mathbb{Q}\}$

3. Let $H_i = \{1, g_i\}, i = 2, 3, 4$. Then $H_i$ is a subgroup of $G$ of order 2. Moreover, $B f_2 = B e_2$ is a center $H_2$-Galois extension of $(B e_2)^{H_2}$ with Galois system $\{b_1 = (1, 0, 0, 0, 0), b_2 = (0, 1, 0, 0, 0); c_1 = (1, 0, 0, 0, 0), c_2 = (0, 1, 0, 0, 0)\}, B f_3 = B e_3$ is a center $H_3$-Galois extension of $(B e_3)^{H_3}$ with Galois system $\{b_1 = (0, 0, 1, 0, 0), b_2 = (0, 0, 0, 1, 0); c_1 = (0, 0, 0, 0, 0)\}$.
1, 0, 0), c_2 = (0, 0, 0, 1, 0)}, and BI_4 = Be_4 is a center H_4-Galois extension of (Be_4)^{H_4} with Galois system \{b_1 = (1, 0, 0, 0, 0), b_2 = (0, 1, 0, 0, 0), b_3 = (0, 0, 1, 0, 0), b_4 = (0, 0, 0, 1, 0); c_1 = (1, 0, 0, 0, 0), c_2 = (0, 1, 0, 0, 0), c_3 = (0, 0, 1, 0, 0), c_4 = (0, 0, 0, 1, 0)\}.

(4) S = \{(0, 0, 0, 0, 0), e_2, e_3, e_4, 1 = (1, 1, 1, 1, 1)\} is the Boolean subalgebra generated by E = \{e_2, e_3, e_4\} in the Boolean algebra of all idempotents in the center of B. The minimal elements in S are f_1 = e_2 and f_2 = e_3, and f_1 \lor f_2 = e_4. We have that Bf_1 = \{(a_1, a_2, 0, 0, 0) | a_1, a_2 \in \mathbb{Q}\}, Bf_2 = \{(0, 0, a_3, a_4, 0) | a_3, a_4 \in \mathbb{Q}\}, and B(1 - f_1 \lor f_2) = \{(0, 0, 0, 0, a_5) | a_5 \in \mathbb{Q}\}. So \(B = Bf_1 \oplus Bf_2 \oplus B(1 - f_1 \lor f_2)\) and Bf_j is a H_j-Galois extension of (Bf_j)^{H_j} for \(j = 1, 2\).

(5) Since \(e_2 = (1, 1, 0, 0, 0), e_3 = (0, 0, 1, 1, 0), \) and \(e_4 = (1, 1, 1, 1, 0)\), we have \(C(1 - e_2) = \{(0, 0, a_3, a_4, a_5) | a_3, a_4, a_5 \in \mathbb{Q}\}\), \(C(1 - e_3) = \{(a_1, a_2, 0, 0, a_5) | a_1, a_2, a_5 \in \mathbb{Q}\}\), and \(C(1 - e_4) = \{(0, 0, 0, 0, a_5) | a_5 \in \mathbb{Q}\}\). So \(g_i\) restricted to \(C(1 - e_i)\) is an identity for each \(g_i \neq 1\) in G.

REFERENCES


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<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>December 1, 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

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