NEW INVERSION FORMULAS FOR THE KRÄTZEL TRANSFORMATION

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(Received 9 June 1999)

ABSTRACT. We study in distributional sense by means of the kernel method an integral transform introduced by Krätzel. It is well known that the cited transform generalizes to the Laplace and Meijer transformation. Properties of analyticity, boundedness, and inversion theorems are established for the generalized transformation.

2000 Mathematics Subject Classification. Primary 44A15, 46F12.

1. Introduction. In this paper, we study the following integral transform,

\[(K_\nu^\rho f)(x) = \int_0^\infty Z_\nu^\rho(xy)f(y)dy, \quad x > 0,\]

(1.1)

where \(Z_\nu^\rho(x)\) denotes the function

\[Z_\nu^\rho(x) = \int_0^\infty t^{\nu-1} e^{-t^{\rho}-x/t} dt\]

(1.2)

with \(\rho > 0, \nu \in \mathbb{N}, \nu \in \mathbb{C}\). The \(K_\nu^\rho\) transform is reduced to the Meijer transform when \(\rho = 1\) and to the Laplace transform when \(\rho = 1\) and \(\nu = \pm 1/2\). Zemanian realized a wide study of the Laplace and Meijer transformations, in distribution spaces (cf. [12, 13]). Later, Krätzel, introduced the \(K_\nu^\rho\) transformation, which generalizes the Meijer transform, and in a series of papers, investigated it in the classical sense (see [5, 6]). In [8], Rao and Debnath investigated the \(K_\nu^\rho\) transformation on certain spaces of distributions by means of the kernel method. Recently, the cited transformation is studied in [1] by the adjoint method on the McBride’s spaces.

In [3], the study of Krätzel is continued obtaining Abelian, Tauberian theorems, and some inversion formulas in the classical sense. Moreover, we can emphasize in [2] the work developed by the authors studying the \(K_\nu^\rho\) transform on weighted \(L^p\) spaces, improving a result of [4].

Motivated by the cited papers, we accomplish a study of the \(K_\nu^\rho\) transform by means of the kernel method on the space of distributions of compact support.

By \(\mathcal{E}(I)\) we denote the infinitely differentiable functions \(\phi(t), \ t \in I = (0, \infty)\), such that for all compact \(K\) we have

\[\gamma_k(\phi) = \sup_{t \in K} \left| \frac{d^k}{dt^k} \phi(t) \right| < \infty,\]

(1.3)

for every \(k \in \mathbb{N}\). By \(\mathcal{E}'(I)\) we denote the dual space of \(\mathcal{E}(I)\). Moreover, by \(\mathcal{D}(I), \mathcal{D}'(I)\) denote the spaces of functions and distributions that can be found in [9, 11].
Consider the following useful properties. In [1] we see that
\[
\frac{d^k}{dx^k} Z_\nu^\rho(x) = (-1)^k Z_\rho^{\nu-k}(x).
\] (1.4)

By (1.4) and the asymptotic behaviour of \( Z_\nu^\rho(x) \), we deduce that for certain positive constant \( \alpha_1 \) and \( k \in \mathbb{N} \), we have
\[
\left| \frac{d^k}{dx^k} (Z_\nu^\rho(x)) \right| \leq \alpha_1 \cdot x^{(2\nu - \rho)/(2\rho + 2)} e^{-x^\rho/(ho + 1)}.
\] (1.5)

Moreover, by [5] we have that if \( \rho \in \mathbb{N} \),
\[
A_{\rho,\nu} Z_\nu^\rho(x) = (-1)^\rho Z_\nu^\rho(x),
\] (1.6)

where
\[
A_{\rho,\nu} = -\rho^{-1} \cdot x^{\nu-\rho+1} \frac{d}{dx} x^{\rho-\nu} \frac{d^\rho}{dx^\rho}.
\] (1.7)

On the other hand, \( B_{\rho,\nu,k} \) denotes
\[
B_{\rho,\nu,k} = C_1 \cdot y^{-\rho k-1} B_{\rho,\nu}^k,
\]

where
\[
B_{\rho,\nu} = x^{-\nu+\rho+1} \frac{d}{dx} x^{\rho+\nu} \frac{d^\rho}{dx^\rho},
\]
\[
C_1 = \frac{(\rho k)^{\rho k-(2\nu-\rho)/(2\rho+2)-1/(\rho+1)+1}}{\Gamma \left( \rho k - \frac{2\nu-\rho}{2\rho+2} - \frac{1}{\rho+1} + 1 \right) \Gamma \left( \frac{\rho k-(2\nu-\rho)/(2\rho+2)-1/(\rho+1)+1}{\rho} + \nu \right)}.
\] (1.8)

The Mellin transform is defined by \( \mathcal{M} \{ f \} (s) = \int_0^\infty x^{s-1} f(x) dx \) and the Mellin transform of the kernel is given by
\[
\mathcal{M} (Z_\nu^\rho(x)) (s) = \frac{1}{\rho} \Gamma(s) \Gamma \left( \frac{s + \nu}{\rho} \right),
\] (1.9)
if \( \text{Re} \ s + \min(0, \text{Re} \nu) > 0 \).

2. The generalized \( K_\nu^\rho \)-transform on \( E'(I) \). Let \( \nu \in \mathbb{C} \) and \( \rho \in \mathbb{N} \). For every \( f \in E'(I) \) we define the generalized \( K_\nu^\rho \) transform by the relation
\[
F(x) = (K_\nu^\rho f)(x) = \langle f(t), Z_\nu^\rho(xt) \rangle.
\] (2.1)

It is easy to see that \( Z_\nu^\rho(xt) \in \mathcal{E}(I) \) for \( x \) fixed, \( x > 0 \). Furthermore, if \( f \) is a locally absolutely integrable function, then the generalized transform of \( f \) is reduced to the classic \( K_\nu^\rho \) transform.

**Proposition 2.1.** The operators \( A_{\rho,\nu} \) and \( B_{\rho,\nu} \) define a continuous linear mapping from \( \mathcal{E}(I) \) into itself.

**Proof.** It is established without difficulty that
\[
y_k(A_{\rho,\nu} \phi) \leq C \cdot \sum_{j=1}^{\rho+1} y_{k+j}(\phi), \quad y_k(B_{\rho,\nu} \phi) \leq C \cdot \sum_{j=1}^{\rho+1} y_{k+j}(\phi),
\] (2.2)
for every \( \phi \in \mathcal{E}(I) \).
We define the generalized operator $A_{\rho,\nu}^*$ on $\mathcal{E}'(I)$ as the adjoint operator of $A_{\rho,\nu}$, that is,
\[
\langle A_{\rho,\nu}^* f, \phi \rangle = \langle f, A_{\rho,\nu} \phi \rangle,
\]
with $f \in \mathcal{E}'(I)$ and $\phi \in \mathcal{E}(I)$. Moreover, by Proposition 2.1, $A_{\rho,\nu}^*$ is a continuous linear mapping from $\mathcal{E}'(I)$ into itself. Note that the same occurs for the operator $B_{\rho,\nu}$.

Next are established some properties of the generalized $K_{\rho}^\nu$ transformation.

**Proposition 2.2.** Let $f \in \mathcal{E}'(I)$. If $F(x)$ denotes the generalized $K_{\rho}^\nu$ transform of $f$, then $F(x)$ is infinitely differentiable on $(0, \infty)$ and
\[
\frac{d^r}{dx^r} F(x) = \left( f(t), \frac{\partial^r}{\partial x^r} Z_{\rho}^\nu(xt) \right), \quad x > 0, \ r \in \mathbb{N}.
\]

**Proof.** Consider $h$ an arbitrary increment in $x > 0$. Assume, without loss of generality, that $0 < \left| h \right| < \left( x/2 \right)^{\rho/(\rho + 1)}$.

It is easy to see that
\[
\frac{F(x + h) - F(h)}{h} = \left( f(t), \frac{1}{h} \left( Z_{\rho}^\nu(t(x + h)) - Z_{\rho}^\nu(tx) \right) \right).
\]

We must see that
\[
\varphi_h(x, t) = \frac{1}{h} \left( Z_{\rho}^\nu(t(x + h)) - Z_{\rho}^\nu(tx) \right) - \frac{\partial}{\partial x} Z_{\rho}^\nu(xt) \to 0 \quad (2.6)
\]
as $h \to 0$, in the sense of the convergence in $\mathcal{E}(I)$, since the result is obtained for $k = 1$, by (2.5) and the continuity of $f(t)$.

For every $r \in \mathbb{N}$ we can write
\[
\frac{\partial^r}{\partial t^r} \varphi_h(x, t) = \frac{1}{h} \int_x^{x + h} \int_x^u \frac{\partial^2}{\partial t^r \partial y^2} \left( \frac{\partial^r}{\partial t^r} Z_{\rho}^\nu(yt) \right) \, dy \, du
\]
\[
= \frac{1}{h} \int_x^{x + h} \int_x^u \frac{\partial^2}{\partial y^2} \left( y^r \frac{d^r}{d(yt)^r} Z_{\rho}^\nu(yt) \right) \, dy \, du
\]
\[
= \frac{1}{h} \int_x^{x + h} \int_x^u \left( r(r - 1)y^{r - 2} \frac{d^r}{d(yt)^r} Z_{\rho}^\nu(yt) + 2ry^{r - 1}t \frac{d^{r+1}}{d(yt)^{r+1}} Z_{\rho}^\nu(yt) \right.
\]
\[
\left. + y^r t^2 \frac{d^{r+2}}{d(yt)^{r+2}} Z_{\rho}^\nu(yt) \right) \, dy \, du.
\]

By virtue of (1.5), given $t \in K$, $K$ compact, $y > x/2$, it follows that there exists a positive constant $C$ such that
\[
\left| \frac{d^{r+j}}{d(yt)^{r+j}} (Z_{\rho}^\nu(yt)) \right| \leq C (yt)^{(2\nu - \rho)/(2\rho + 2)} \cdot e^{-(tx/2)^{\rho/(\rho + 1)}}, \text{ for } j \in \mathbb{N}.
\]

Therefore
\[
\left| \frac{\partial^r}{\partial t^r} \varphi_h(x, t) \right| \leq C \cdot t^{(2\nu - \rho)/(2\rho + 2)} \cdot e^{-(tx/2)^{\rho/(\rho + 1)}} \cdot \frac{1}{h} \int_x^{x + h} \int_x^u (y^{r - 2} + y^{r - 1}t + y^r t^2) \, dy \, du
\]
\[
\leq C \cdot e^{-\alpha(tx)^{\rho/(\rho + 1)}}
\]
for all $t \in K$ and $0 < \left| h \right| < (x/2)^{\rho/(\rho + 1)}$, being $\alpha$ a suitable constant.
Then given $\epsilon > 0$, there exists $t_0 \in K = [a, b]$ such that
\[
\left| \frac{\partial^r}{\partial t^r} \varphi_h(x, t) \right| < \epsilon \quad (2.10)
\]
for $t > t_0$ and $0 < |h| < x^{n/(\rho + 1)}/2$.

Furthermore, for each $t \in [a, t_0]$ we have
\[
\left| \frac{\partial^r}{\partial t^r} \varphi_h(x, t) \right| \leq C \cdot \left| \frac{1}{h} \int_x^{x+h} \int_x^u (y^{r-2} + y^{r-1} t + y^r t^2) dy du \right|. \quad (2.11)
\]

Then $(\partial^r / \partial t^r) \varphi_h(x, t) \to 0$ as $h \to 0$ uniformly in $t \in [a, t_0]$. Therefore it is concluded that $\gamma_r(\varphi_h(x, t)) \to 0$ as $h \to 0$.

By proceeding inductively the result follows.

**Proposition 2.3.** Let $f \in \mathcal{E}'(I)$ and $F(x) = (K^\rho_\nu f)(x)$ for $x > 0$, then
\[
|F(x)| \leq C \cdot x^{r+(2\nu-\rho)/(\rho+2)} \cdot e^{-\alpha x^{\rho/(\rho+1)}}, \quad x > 0, \quad (2.12)
\]
being $C$ and $\alpha$ suitable constants and $r \in \mathbb{N}$.

**Proof.** We know by [13, Theorem 1.8-1, pages 18–19] that there exists $r \in \mathbb{N}$ such that
\[
|F(x)| \leq C \max_{0 \leq k \leq r} \gamma_k(Z^\nu_\rho(x^t)) \quad \forall x > 0. \quad (2.13)
\]

By the asymptotic behaviour we have for every $K$ compact,
\[
|F(x)| \leq C \max_{0 \leq k \leq r} \sup_{t \in K} \left| x^k \frac{d^k}{d(x^t)^k} (Z^\nu_\rho(x^t)) \right|
\leq C \max_{0 \leq k \leq r} \sup_{t \in K} \left| x^k (x^t)^{(2\nu-\rho)/(\rho+2)} e^{-(x^t)^{\rho/(\rho+1)}} \right|
\leq C \cdot x^{r+(2\nu-\rho)/(\rho+2)} \cdot \max_{0 \leq k \leq r} \sup_{t \in K} \left| t^{(2\nu-\rho)/(\rho+2)} e^{-(x^t)^{\rho/(\rho+1)}} \right|
\leq C \cdot x^{r+(2\nu-\rho)/(\rho+2)} e^{-\alpha x^{\rho/(\rho+1)}}. \quad \Box
\]

With the following proposition we obtain an operational formula for the generalized $K^\rho_\nu$ transform, that includes the operator $A^*_{p,\nu,\rho} \in \mathbb{N}$.

**Proposition 2.4.** Let $P$ be a polynomial, if $f \in \mathcal{E}'(I)$, then
\[
(K^\rho_\nu P(A^*_{p,\nu,\rho}) f)(x) = P((-x)^p) (K^\rho_\nu f)(x) \quad (2.15)
\]
for $x > 0$ and $\rho \in \mathbb{N}$.

**Proof.** By (1.6) and according to Proposition 2.1 it follows that
\[
(K^\rho_\nu P(A^*_{p,\nu,\rho}) f)(x) = \langle P(A^*_{p,\nu,\rho}) f(t), Z^\nu_\rho(x^t) \rangle
= \langle f(t), P(A_{p,\nu,\rho}) Z^\nu_\rho(x^t) \rangle
= \langle f(t), P((-x)^p) Z^\nu_\rho(x^t) \rangle
= P((-x)^p) (K^\rho_\nu f)(x). \quad \Box
\]
Now we establish an inversion theorem for the $K_\rho^\nu$-transform using a similar procedure to employ by Malgonde and Saxena [7].

**Lemma 2.5.** Let $f \in E'(I)$. Then

$$\int_0^\infty x^{-s} \langle f(t), Z_\rho^\nu(xt) \rangle \, dx = \left\langle f(t), \int_0^\infty x^{-s} Z_\rho^\nu(xt) \, dx \right\rangle. \quad (2.17)$$

**Proof.** Let $N \in \mathbb{N}$. First, we prove that

$$\int_0^N x^{-s} \langle f(t), Z_\rho^\nu(xt) \rangle \, dx = \left\langle f(t), \int_0^N x^{-s} Z_\rho^\nu(xt) \, dx \right\rangle. \quad (2.18)$$

We take $\{x_{r,l}\}_{r=0}^l$ a partition of the interval $[0,N]$ such that $d_l = x_{r,l} - x_{r-1,l}$ for each $r = 1, 2, \ldots, l$. Then we can write

$$\int_0^N x^{-s} \langle f(t), Z_\rho^\nu(xt) \rangle \, dx = \lim_{l \to \infty} d_l \cdot \sum_{r=0}^l x_{r,l}^{-s} \langle f(t), Z_\rho^\nu(x_{r,l}t) \rangle. \quad (2.19)$$

Then (2.18) is established if we demonstrate

$$\lim_{l \to \infty} d_l \cdot \sum_{r=0}^l x_{r,l}^{-s} \cdot Z_\rho^\nu(x_{r,l}t) = \int_0^N x^{-s} Z_\rho^\nu(xt) \, dx \quad (2.20)$$

in the sense of the convergence in $E(I)$.

Then, since the function

$$g(t,x) = \begin{cases} x^{-s+k} \frac{d^k}{d(xt)^k}(Z_\rho^\nu(xt)) & \text{if } x \in [0,N], \\ 0 & \text{if } x = 0, \end{cases} \quad (2.21)$$

with $t \in [a,b]$, $0 < a < b < \infty$, is uniformly continuous on $(t,x) \in [a,b] \times [0,N]$, we have that if $\epsilon > 0$ and $m \in \mathbb{N}$, there exists $l_0 \in \mathbb{N}$ such that

$$\sup_{t \in K} \left| \frac{d^k}{dtk} \left[ d_l \cdot \sum_{r=0}^l x_{r,l}^{-s} Z_\rho^\nu(x_{r,l}t) - \int_0^N x^{-s} Z_\rho^\nu(xt) \, dx \right] \right| < \epsilon \quad (2.22)$$

for $l > l_0$. Therefore we obtain (2.20).

Using (1.5), for $m, k \in \mathbb{N}$, we have that

$$\left| \frac{d^k}{dtk} \int_N^\infty x^{-s} Z_\rho^\nu(xt) \, dx \right| \leq C \cdot t^{(2\nu-\rho)/(2p+2)} \int_N^\infty x^{-\Re s+(2\nu-\rho)/(2p+2)} e^{-\alpha x^{p/(p+1)}} \, dx \to 0 \quad (2.23)$$

uniformly in $t \in K$, as $N \to \infty$.

Moreover, by Proposition 2.3 we have

$$\left| \int_N^\infty x^{-s} \langle f(t), Z_\rho^\nu(xt) \rangle \, dx \right| \leq C \int_N^\infty x^{-s+\nu+(2\nu-\rho)/(2p+2)} e^{-\alpha x^{p/(p+1)}} \, dx, \quad (2.24)$$

being $\alpha$ a suitable constant.
Then
\[
\lim_{N \to \infty} \int_{-N}^{N} x^{-s} \langle f(t), Z_{\rho}^{\nu}(xt) \rangle \, dx = 0.
\]  
(2.25)

Hence by (2.18), (2.23), and (2.25) we obtain (2.17).

The following lemma is obtained from [7, Lemmas 2 and 3] with a slight modification.

**Lemma 2.6.** Let \( \phi \in D(I) \), we denote
\[
\psi(s) = \int_{0}^{\infty} y^{-s} \phi(y) \, dy.
\]  
(2.26)

Then
\[\int_{-R}^{R} (f(u), u^{\sigma+iw-1}\psi(\sigma+iw) \, dw = (f(u), \int_{-R}^{R} u^{\sigma+iw-1}\psi(\sigma+iw) \, dw) \quad \text{with} \quad \sigma > 0 \quad \text{and} \quad R > 0, \]
(iii) \((1/\pi) \int_{0}^{\infty} \phi(y)(u/y)^{\sigma}(\sin(R \log(u/y))/u \log(u/y)) \, dy \to \phi(u) \quad \text{as} \quad R \to \infty, \quad \text{in the sense of the convergence in} \quad \mathcal{E}(I), \quad \text{with} \quad \sigma > 0.\

Now we demonstrate the first inversion theorem.

**Theorem 2.7.** Let \( f \in \mathcal{E}'(I) \), \( \phi \in \mathbb{D}(I) \), \( \nu \in \mathbb{C} \) and \( \sigma < 1 + \min(0, \text{Re } \nu) \). Then
\[
\lim_{R \to \infty} \left\langle \frac{1}{2\pi i} \int_{\sigma-iR}^{\sigma+iR} \frac{1}{K(s)} y^{-s} \int_{0}^{\infty} x^{-s} F(x) \, dx \, ds, \phi(y) \right\rangle = \langle f(t), \phi(t) \rangle,
\]  
(2.27)

where \( F(x) = (K_{\nu}^{\rho} f)(x) \), for \( x > 0 \), and
\[
K(s) = \frac{1}{\rho} \Gamma(1-s) \Gamma \left( \frac{1-s+\nu}{\rho} \right).
\]  
(2.28)

**Proof.** Given \( f \in \mathcal{E}'(I) \), we denote \( F(x) = \langle f(t), Z_{\rho}^{\nu}(xt) \rangle \), for \( x > 0 \). It is easy to see that the function
\[
\varphi_{R}(y) = \frac{1}{2\pi i} \int_{\sigma-iR}^{\sigma+iR} \frac{1}{K(s)} y^{-s} \int_{0}^{\infty} x^{-s} F(x) \, dx \, ds
\]  
(2.29)

is continuous for \( y > 0, \ R > 0 \). Therefore, \( \varphi_{R}(y) \) defines a regular distribution in \( \mathbb{D}'(I) \) being
\[
\langle \varphi_{R}(y), \phi(y) \rangle = \frac{1}{2\pi i} \int_{0}^{\infty} \phi(y) \int_{\sigma-iR}^{\sigma+iR} \frac{1}{K(s)} y^{-s} \int_{0}^{\infty} x^{-s} F(x) \, dx \, ds \, dy
\]  
(2.30)

for all \( \phi \in \mathbb{D}(I) \).

By the Fubini’s theorem we can interchange the order of integration and we can write
\[
\langle \varphi_{R}(y), \phi(y) \rangle = \frac{1}{2\pi i} \int_{\sigma-iR}^{\sigma+iR} \frac{1}{K(s)} \left[ \int_{0}^{\infty} x^{-s} \langle f(t), Z_{\rho}^{\nu}(xt) \rangle \, dx \right] \int_{0}^{\infty} y^{-s} \phi(y) \, dy \, ds
\]  
(2.31)
By Lemma 2.5, we get
\[
\langle \varphi_R(y), \phi(y) \rangle = \frac{1}{2\pi i} \int_{\sigma - iR}^{\sigma + iR} \frac{1}{K(s)} \int_0^\infty u^{-s} Z_\rho^\nu(u) \, du \int_0^\infty y^{-s} \phi(y) \, dy \, ds
\]
\[
= \frac{1}{2\pi i} \int_{\sigma - iR}^{\sigma + iR} (f(t), t^{-s-1}) \int_0^\infty y^{-s} \phi(y) \, dy \, ds.
\]
(2.32)

Then Lemma 2.6(i) permits us to obtain
\[
\langle \varphi_R(y), \phi(y) \rangle = \langle f(t), \frac{1}{2\pi i} \int_{\sigma - iR}^{\sigma + iR} t^{-s} \phi(y) \, dy \, ds \rangle.
\]
(2.33)

Finally, interchanging the order of integration and using Lemma 2.6(ii) we achieve
\[
\langle \varphi_R(y), \phi(y) \rangle = \langle f(t), \frac{1}{\pi} \int_0^\infty \left( \frac{u}{y} \right)^\sigma \phi(y) \sin \left( R \log \left( \frac{u}{y} \right) \right) \frac{u \log \left( \frac{u}{y} \right)}{u log (u/y)} \, dy \rangle \rightarrow \langle f(t), \phi(t) \rangle
\]
\[
(2.34)
\]
as \( R \to \infty \). With this, Theorem 2.7 is demonstrated.

By Theorem 2.7 the following uniqueness theorem is deduced.

**Theorem 2.8.** Let \( f, g \in \mathcal{E}'(I) \) and \( \nu \in \mathbb{C} \). If \( (K_\rho^\nu f)(x) = (K_\rho^\nu g)(x) \), for \( x > 0 \), then \( f = g \) in the sense of an equality in \( \mathcal{D}'(I) \).

**Proof.** It is sufficient to see that for each \( \phi \in \mathcal{D}(I) \), \( \langle f(t) - g(t), \phi(t) \rangle = 0 \).

This fact is established immediately since,
\[
\langle f(t) - g(t), \phi(t) \rangle = \left\langle \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\sigma - iR}^{\sigma + iR} t^{-s} \int_0^\infty y^{-s} \phi(y) \, dy \, ds, \phi(y) \right\rangle
\]
\[
= \left\langle \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\sigma - iR}^{\sigma + iR} \frac{1}{K(s)} \int_0^\infty x^{-s} \left( (K_\rho^\nu f)(x) - (K_\rho^\nu g)(x) \right) \, dx \, ds, \phi(y) \right\rangle = 0,
\]
(2.35)

where \( \sigma \) and \( K(s) \) are as in Theorem 2.7.

In the following theorem we obtain another inversion formula, in which appears a generalization of an operator of Post-Widder type, being obtained as a particular case an inversion formula for the Laplace and Meijer transformation.

**Theorem 2.9.** Let \( f \in \mathcal{E}'(I) \) and \( \text{Re} \nu \geq \rho/2 - 1 \). If \( F(x) \) denotes the \( K_\rho^\nu \) generalized transform of \( f \), then
\[
\lim_{k \to \infty} \left\langle B_{\rho,\nu,k}^* F \left( \frac{\rho k}{y} \right), \phi(y) \right\rangle = \langle f(t), \phi(t) \rangle
\]
(2.36)

for each \( \phi \in \mathcal{D}(I) \), where \( B_{\rho,\nu,k}^* \) is the adjoint operator of \( B_{\rho,\nu,k} \).

**Proof.** Let \( f \in \mathcal{E}'(I) \) and \( F(x) = (f(t), Z_\rho^\nu(xt)) \), for \( x > 0 \).

By Proposition 2.1 and (1.6) we have
\[
B_{\rho,\nu,k}^* F \left( \frac{\rho k}{y} \right) = C_1 y^{-\rho k - 1} \left\langle f(t), B_{\nu,\rho,k}^* Z_\rho^\nu \left( \frac{\rho k t}{y} \right) \right\rangle
\]
\[
= C_1 y^{-\rho k - 1} \left\langle f(t), \rho^k (\rho k t)^{\rho k} Z_\rho^\nu \left( \frac{\rho k t}{y} \right) \right\rangle,
\]
(2.37)

\( C_1 \) being introduced in Section 1.
Moreover, $B_{\rho,\nu,k}^* F(\rho k / y)$ defines a regular distribution in $\mathcal{D}'(I)$ by
\[
\left\langle B_{\rho,\nu,k}^* F \left( \frac{\rho k}{y} \right), \phi(y) \right\rangle = \int_0^\infty B_{\rho,\nu,k}^* F \left( \frac{\rho k}{y} \right) \phi(y) \, dy, \quad \forall \phi \in \mathcal{D}(I).
\] (2.38)

We must see that
\[
\left\langle B_{\rho,\nu,k}^* F \left( \frac{\rho k}{y} \right), \phi(y) \right\rangle = \left\langle f(t), C_1 t^\rho \int_0^\infty y^{-\rho k - 1} Z_\rho^y \left( \frac{\rho k t}{y} \right) \phi(y) \, dy \right\rangle
\] (2.39)

for every $\phi \in \mathcal{D}(I)$.

Consider $0 < a < b < \infty$ such that $\phi(x) = 0, x \notin [a, b], \phi \in \mathcal{D}(I)$.

We denote by $\{y_{m,l}\}_{m=0}^l$ a partition of $[a, b]$ with $d_l = y_{m,l} - y_{m,l-1}, m = 1, 2, \ldots, l$, then we achieve
\[
\int_0^\infty B_{\rho,\nu,k}^* F \left( \frac{\rho k}{y} \right) \phi(y) \, dy = \lim_{l \to \infty} d_l \sum_{m=0}^l y_{m,l}^{-\rho k - 1} Z_\rho^y \left( \frac{\rho k t}{y_{m,l}} \right) \phi(y_{m,l}) = \lim_{l \to \infty} \left\langle f(t), t^\rho C_1 \int_0^\infty y^{-\rho k - 1} Z_\rho^y \left( \frac{\rho k t}{y} \right) \phi(y) \, dy \right\rangle
\] (2.40)

Therefore we must see that
\[
\lim_{l \to \infty} d_l \sum_{m=0}^l y_{m,l}^{-\rho k - 1} Z_\rho^y \left( \frac{\rho k t}{y_{m,l}} \right) \phi(y_{m,l}) = \int_a^b y^{-\rho k - 1} Z_\rho^y \left( \frac{\rho k t}{y} \right) \phi(y) \, dy
\] (2.41)
in the sense of the convergence in $\mathcal{E}(I)$.

Consider $K \subset (0, \infty)$ compact and $r \in \mathbb{N}$.

Then we have
\[
\frac{d^r}{dt^r} \left( d_l \sum_{m=0}^l y_{m,l}^{-\rho k - 1} Z_\rho^y \left( \frac{\rho k t}{y_{m,l}} \right) \phi(y_{m,l}) - \int_a^b y^{-\rho k - 1} Z_\rho^y \left( \frac{\rho k t}{y} \right) \phi(y) \, dy \right)
\] (2.42)

Hence, since the function $y^{-\rho k - 1} (d^r/dt^r) (Z_\rho^y (\rho k t / y)) \phi(y)$ is uniformly continuous for $(x, y) \in [a, b] \times K$, then
\[
\lim_{l \to \infty} d_l \sum_{m=0}^l y_{m,l}^{-\rho k - 1} Z_\rho^y \left( \frac{\rho k t}{y_{m,l}} \right) \phi(y_{m,l}) = \int_a^b y^{-\rho k - 1} Z_\rho^y \left( \frac{\rho k t}{y} \right) \phi(y) \, dy
\] (2.43)

uniformly in $y \in K$. Thus (2.39) is demonstrated.
On the other hand, making a change of variables we obtain

$$\int_0^\infty B^*_{\rho,\nu,k} F \left( \frac{\rho k}{\gamma} \right) \phi(y) \, dy = \left( \frac{g(t)}{t}, C_1 t^{-\rho k-1} \int_0^\infty u^{\rho k} Z_\rho^\nu \left( \frac{\rho ku}{t} \right) \psi(u) \, du \right)$$  \hspace{1cm} (2.44)

with \(g(t) = (1/t)f(1/t)\) and \(\psi(u) = (1/u)\phi(1/u)\).

To complete the demonstration we see that

$$\lim_{k \to \infty} C_1 t^{-\rho k-1} \int_0^\infty u^{\rho k} Z_\rho^\nu \left( \frac{\rho ku}{t} \right) \psi(u) \, du = \psi(t)$$  \hspace{1cm} (2.45)

in the sense of the convergence in \(E(I)\).

Now, according to (1.9) we have

$$\int_0^\infty u^{\rho k-(2\nu-\rho)/(2\nu+2)-(1/(\rho+1))} Z_\rho^\nu \left( \frac{\rho ku}{t} \right) \, du$$

$$= \left( \frac{t}{\rho k} \right)^{(\rho+1)k-(2\nu-\rho)/(2\nu+2)-(1/(\rho+1))+1} \rho^{-1} \Gamma \left( \frac{\rho k - 2\nu - \rho}{2\nu + 2} - \frac{1}{\rho+1} + 1 \right) \cdot \Gamma \left( \frac{\rho k - (2\nu - \rho)/(2\nu+2) - (1/(\rho+1)) + 1 + \nu}{\rho} \right).$$  \hspace{1cm} (2.46)

Then, we can deduce that

$$\frac{dr}{dt} \int_0^\infty u^{(2\nu-\rho)/(2\nu+2)+(1/(\rho+1))} \left( \frac{u}{t} \frac{d}{du} \right)^r \psi(u) - t^{(2\nu-\rho)/(2\nu+2)+(1/(\rho+1))} \frac{dr}{dt} \psi(t) \, du$$  \hspace{1cm} (2.47)

for every \(r \in \mathbb{N}\).

Therefore by (1.5) we have that for all \(t \in K, K \subset (0, \infty), K \text{ compact}, \) there exists a constant \(C_0 > 0\) such that

$$\left| \frac{dr}{dt} \left( C \cdot t^{-(\rho+1)k-1} \int_0^\infty Z_\rho^\nu \left( \frac{\rho kt}{y} \right) z^{(\rho+1)k} \psi(z) \, dz - \psi(t) \right) \right|$$

$$\leq C \left( \frac{(\rho k)^{\rho k-\rho+1-1/(\rho+1)}}{(\rho+1)k)!} \right) t^{-\rho k-\rho/(\rho+1)} \frac{\rho}{\rho+1} \int_0^\infty z^{(\rho+1)k} e^{-\alpha z} \, dz \cdot \int \psi(z) \, dz.$$  \hspace{1cm} (2.48)

We divided the last integral in this way

$$C_1 \int_0^\infty = C_1 \left( \int_0^{((\rho+1)k)^{1/(\rho+1)}} + \int_{((\rho+1)k)^{1/(\rho+1)}}^{((\rho+4)k)^{1/(\rho+1)}} \int_{((\rho+4)k)^{1/(\rho+1)}}^{((\rho+4)k)^{1/(\rho+1)}} \frac{dr}{dt} \psi(t) \, dz \right)$$

$$= I_1(t,k) + I_2(t,k) + I_3(t,k)$$  \hspace{1cm} (2.49)

with \(\eta > 0\) and \(C_1 = C((\rho k)^{\rho k-\rho+1-1/(\rho+1)} t^{-\rho k-\rho/(\rho+1)} \rho / (\rho+1)).\)
For each $t \in K$, we obtain
\[
|I_1(t,k)| \\
\leq C \frac{(\rho k)^{\rho k - 1} - 1}{(\rho + 1)k!} \frac{\rho}{\rho + 1} t^{-\rho k - \rho/(\rho + 1)} \cdot \left( \int_0^{((\rho + 1)k)^{1/\rho} (1 - \eta)^{1/\rho} t(k)^{-1}} u^{\rho k + (2\nu - \rho)/(2\rho + 2)} e^{-(\rho k u/t)\rho/(\rho + 1)} \left| \left( \frac{u}{t} \frac{d}{du} \right)^r \psi(u) \right| du \\
+ \int_0^{((\rho + 1)k)^{1/\rho} (1 - \eta)^{1/\rho} t(k)^{-1}} u^{\rho k - 1}/(\rho + 1) e^{-(\rho k u/t)\rho/(\rho + 1)} \left| \frac{dr}{dt} \psi(t) \right| du \right)
\]
\[
\leq C |t - u| \frac{(\rho + 1)k}{(\rho + 1)k!} \int_0^{1 - \eta} x^{(\rho + 1)k} e^{-(\rho + 1)kx} dx,
\]
(2.50)
and making an appropriate change of variables, we obtain
\[
|I_1(t,k)| \leq C \frac{(\rho + 1)k}{(\rho + 1)k!} \int_0^{1 - \eta} x^{(\rho + 1)k} e^{-(\rho + 1)kx} dx,
\]
(2.51)
\[C\] being a suitable constant and $\text{Re} \nu \geq \rho/2 - 1$.
Now, using [10, Theorem (5b), page 288] we achieve
\[
I_1(t,k) \to 0,
\]
(2.52)
as $k \to \infty$ uniformly in $t \in K$.
Proceeding in a similar way we obtain that
\[
I_3(t,k) \to 0,
\]
(2.53)
as $k \to \infty$ uniformly in $t \in K$.
Then, it remains $I_2(t,k)$, for this, using the mean value theorem we can write
\[
\left| u^{(2\nu - \rho)/(2\rho + 2) + 1/(\rho + 1)} \left( \frac{u}{t} \frac{d}{du} \right)^r \psi(u) - t^{(2\nu - \rho)/(2\rho + 2) + 1/(\rho + 1)} \frac{dr}{dt} \psi(t) \right| \leq C_2 |t - u|
\]
(2.54)
for $u, t \in (0, \infty)$ and $C_2$ a suitable constant.
Then if $u \in ((\rho + 1)k)^{1/\rho} (1 - \eta)^{1/\rho} t(k)^{-1}$, $((\rho + 1)k)^{1/\rho} (1 + \eta)^{1/\rho} t(k)^{-1}$ we have
\[
|I_2(t,k)| \leq C_3 \frac{(\rho + 1)k}{(\rho + 1)k!} \int_0^{1 + \eta} x^{(\rho + 1)k} e^{-(\rho + 1)kx} dx,
\]
(2.55)
applying [10, Theorem (5b), page 287] we obtain the desired result.
Therefore we can deduce that
\[
\lim_{k \to \infty} \langle C_{\rho,\nu,k} F \left( \frac{\rho k}{Y} \right), \phi(y) \rangle = \langle g(t), \psi(t) \rangle = \langle f(t), \phi(t) \rangle \quad \forall \phi \in D(I).
\]
(2.56)
References


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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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