ELEMENTS IN EXCHANGE RINGS WITH RELATED COMPARABILITY

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Abstract. We show that if $R$ is an exchange ring, then the following are equivalent: (1) $R$ satisfies related comparability. (2) Given $a, b, d \in R$ with $aR + bR = dR$, there exists a related unit $w \in R$ such that $a + bw = dw$. (3) Given $a, b \in R$ with $aR = bR$, there exists a related unit $w \in R$ such that $a = bw$. Moreover, we investigate the dual problems for rings which are quasi-injective as right modules.

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Let $R$ be an associative ring with identity. From [6], $R$ is said to satisfy related comparability provided that for any idempotents $e, f \in R$ with $e = 1 + ab$ and $f = 1 + ba$ for some $a, b \in R$, there exists a $u \in B(R)$ such that $ueR \leq eR$ and $(1 - u)fR \leq (1 - u)eR$. The class of rings satisfying related comparability is quite large. It includes regular rings satisfying general comparability [10], one-sided unit regular rings [8] and partially unit-regular rings, while there still exist rings satisfying related comparability, which belong to none of the above classes (cf., [7, Example 10]).

In [4, 5], we studied related comparability over regular rings. In [6, 7], we investigated related comparability over exchange rings. It is shown that every exchange ring satisfying related comparability is separative [1]. Also, we show that related comparability over exchange rings is a Morita invariant. $R$ is said to be an exchange ring if for every right $R$-module $A$ and any two decompositions $A = M \oplus N = \bigoplus_{i \in I} A_i$, where $M \cong R$ and the index set $I$ is finite, then there exist submodules $A'_i \subseteq A_i$ such that $A = M \oplus (\bigoplus_{i \in I} A'_i)$. Many authors have investigated exchange rings with some kind of comparability properties so as to study problems related partial cancellation properties of modules (see [1, 2, 6, 7, 12, 13]).

In this paper, we investigate related comparability over exchange rings by related units. Recall that $w \in R$ is said to be a related unit of $R$ if there exists some $e \in B(R)$ such that $w = eu + (1 - e)v$ for some $u, v \in R$, where $eu$ is right invertible in $eR$ and $(1 - e)v$ is left invertible in $(1 - e)R$. $w \in R$ is said to be a semi-related unit of $R$ if $w \in R$ is a related unit modulo $J(R)$. By virtue of semi-related units, we also give some new element-wise properties of rings which are quasi-injective as right modules.

Throughout, all rings are associative with identities. $B(R)$ denotes the set of all central idempotents of $R$ and $r \cdot \text{ann}(b)(1 \cdot \text{ann}(b))$ denotes the right (left) annihilator of $b \in R$. 
**Lemma 1.** Let $R$ be an exchange ring. Then $R$ satisfies related comparability if and only if so does the opposite ring $R^{\text{op}}$ of $R$.

**Proof.** Since $R$ is an exchange ring, by virtue of [11, Proposition], so is the opposite ring $R^{\text{op}}$ of $R$. Assume that $R$ satisfies related comparability. Given $a^{\text{op}}, b^{\text{op}} \in R^{\text{op}}$ with $a^{\text{op}}x^{\text{op}} + b^{\text{op}} = 1^{\text{op}}$, then we have $xa + b = 1$ in $R$. In view of [6, Theorem 4], there exists a $y \in R$ such that $x + by$ is a related unit of $R$. Thus, we have some $e \in B(R)$ such that $(x + by)e$ is right invertible in $eR$ and $(x + by)(1 - e)$ is left invertible in $(1 - e)R$. By [5, Lemma 4], we claim that there are $z_1, z_2 \in R$ such that $(a + z_1 b)e$ is left invertible in $eR$ and $(a + z_2 b)(1 - e)$ is right invertible in $(1 - e)R$. Let $z = z_1 + z_2 (1 - e)$. Then $a + zb$ is a related unit of $R$. Consequently, $a^{\text{op}} + b^{\text{op}}e^{\text{op}}$ is a related unit of $R^{\text{op}}$. By [6, Theorem 4], we conclude that $R^{\text{op}}$ satisfies related comparability. The converse is clear from $R \cong (R^{\text{op}})^{\text{op}}$.

**Theorem 2.** Let $R$ be an exchange ring. Then the following are equivalent:

1. $R$ satisfies related comparability.
2. Given $a, b, d \in R$ with $aR + bR = dR$, there exists a related unit $w \in R$ such that $a + bt = dw$.
3. Given $a, b \in R$ with $aR = bR$, there exists a related unit $w \in R$ such that $a = bw$.
4. Given $a, b, d \in R$ with $Ra + Rb = Rd$, there exists a related unit $w \in R$ such that $a + tb = wd$.
5. Given $a, b \in R$ with $Ra = Rb$, there exists a related unit $w \in R$ such that $a = wb$.

**Proof.** (2)$\Rightarrow$(1). Trivial from [6, Theorem 4].

(1)$\Rightarrow$(2). Given $a, b, d \in R$ with $aR + bR = dR$. Let $g : dR \to dR/bR$ be the canonical map, $f_1 : R \to aR$ given by $r \to ar$ for any $r \in R$, and $f_2 : R \to bR$ given by $r \to br$ for any $r \in R$. Since $aR + bR = dR$, we know that $g f_1, g f_2$ are epimorphisms. On the other hand, $R$ is a projective $R$-module. So there is some $\psi \in \text{End}_R R$ such that $g f_1 = g f_2 \psi$. Since $g f_1$ is an epimorphism, we also have some $\psi \in \text{End}_R R$ such that $g f_2 \psi = g f_3$. From $\alpha \psi + (1 - \alpha \psi) = 1$, there is a $y \in \text{End}_R R$ such that $\alpha + (1 - \alpha \psi)y = w$ is a related unit of $\text{End}_R R$. Therefore, we see that $g f_1 = g f_3 \alpha = g f_3 (\alpha + (1 - \alpha \psi)y) = g f_3 w$, and then $g (f_1 - f_3 w) = 0$. Thus, we have $\text{Im}(f_1 - f_3 w) \subseteq \text{Ker}g = bR$. By the projectivity of right $R$-module $R$, there exists some $B \in \text{End}_R R$ such that $f_2 B = f_1 - f_3 w$. Therefore, we claim that $a + bB(1) = f_1(1) + f_2(1)B(1) = f_3(1)w(1) = dw(1)$. It is easy to verify that $w(1)$ is a related unit of $R$.

(1)$\Rightarrow$(3). Given $a, b \in R$ with $aR = bR$, there exist $s, t \in R$ such that $a = bs$ and $b = at$. Thus, $b = bst$. Since $st + (1 - st) = 1$, by virtue of [6, Theorem 4], there exists some $z \in R$ such that $s + (1 - st)z = w$ is a related unit of $R$. Hence $a = bs = b(s + (1 - st)z) = bw$, as desired.

(3)$\Rightarrow$(1). Given any regular $a \in R$. Then there exists some $b \in R$ such that $a = aba$, so $aR = abR$. Thus $a = abw$ for some related unit $w \in R$. Since $ab + (1 - ab) = 1$, we see that $a + (1 - ab)w = (ab + (1 - ab))w = w$. By [5, Lemma 4], there is some $z \in R$ such that $b + z(1 - ab) = m$ is a related unit of $R$. Hence $a = aba = a(b + z(1 - ab))a = ama$. According to [6, Theorem 2], we claim that $R$ satisfies related comparability.

(1)$\iff$(4)$\iff$(5). By [11, Proposition], we see that the opposite ring $R^{\text{op}}$ of $R$ is
Thus we can find some \( k \) then \( y \), then \( \text{Theorem 4} \), we can find a check that \( ky \) is a related unit.

Clearly, \( a \) is a related unit-regular, as asserted.

\begin{corollary}
Let \( R \) be an exchange ring. Then the following are equivalent:
\begin{enumerate}
\item \( R \) satisfies related comparability.
\item Given \( a, b \in R \) with \( aR + r \cdot \text{ann}(b) = R \), there exists some \( k \in r \cdot \text{ann}(b) \) such that \( ax + k = 1 \). Since \( R \) satisfies related comparability, by virtue of [6, Theorem 4], we can find a \( y \in R \) such that \( a + ky \) is a related unit of \( R \). It is easy to check that \( ky \) is a related unit-regular, as asserted.
\item Given \( a, b \in R \) with \( R \), there exist some \( s, t \in R \) such that \( a = bs \) and \( b = at \). Obviously, \( 1 - st \in r \cdot \text{ann}(b) \). Since \( st + (1 - st) = 1 \), we have \( sR + r \cdot \text{ann}(b) = R \). Thus we can find some \( k \in r \cdot \text{ann}(b) \) such that \( s + k = w \) is a related unit of \( R \), and then \( a = bs = b(s + k) = bw \), as asserted.
\item \( R \) has the finite stable range. Then the following are equivalent:
\item \( R \) has the finite stable range.
\item Given \( a, b, d \in R \) with \( dR \) there exist some related unit-regular \( w_1, w_2 \in R \) such that \( aw_1 + bw_2 = d \).
\item Given \( a, b, d \in R \) with \( Ra + Rb = Rd \), there exist some related unit-regular \( w_1, w_2 \in R \) such that \( w_1a + w_2b = d \).
\end{enumerate}
\end{corollary}

\textbf{PROOF.} (1)\( \Rightarrow \)(2). Given \( a, b, d \in R \) with \( aR + bR = dR \), there exist some related unit-regular \( w_1, w_2 \in R \). For right \( R \)-module \( R^2 \), the two sets \( \{a, b\} \) and \( \{0, d\} \) generate the same right \( R \)-submodule of \( R^2 \). Thus, we can find \( A, B \in M_2(R) \) such that \( (a, b) = (0, d)A \). Assume that \( A = (a_{ij}), B = (b_{ij}), I_2 = AB = (c_{ij}) \in M_2(R) \). Since \( AB = (I_2 - AB) = I_2 \), we have \( (a_{21}, a_{22})(b_{12}, b_{22})^T + c_{22} = 1 \). Since \( R \) is an exchange ring satisfying related comparability, its stable range can only be \( 1, 2 \) or \( \infty \) by [7, Theorem 3]. So \( 2 \) is in the stable range of \( R \). Thus, we have some \( (y_1, y_2) \in R^2 \) such that \( (a_{21}, a_{22}) + c_{22}(y_1, y_2) \in R^2 \) is unimodular. Set \( Y = \left( \begin{array}{cc}
0 & 0 \\
y_1 & y_2
\end{array} \right) \). Then, we claim that the second row of \( A + (I_2 - AB)Y = U \) is unimodular. Clearly, \( (0, d)U = (0, d)A = (a, b) \). Since \( u_{21}R + u_{22}R = R \), we can find orthogonal idempotents \( e_1 \in u_{21}R, e_2 \in u_{22}R \) such that \( e_1 + e_2 = 1 \). Assume that \( e_1 = u_{21}x_1, e_2 = u_{22}x_2 \). Let \( w_1 = x_1e_1, w_2 = x_2e_2 \). Then \( w_1 \) and \( w_2 \) are both regular in \( R \). Moreover, we have \( u_{21}w_1 + u_{22}w_2 = 1 \). By the related comparability of \( R \), we claim that both \( w_i \) are related unit-regular, as asserted.
(2) ⇒ (1). Given any regular \( x \in R \). Then \( x = x y x \) for a \( y \in R \). So we have \( xR + (1 - xy)R = R \), and then \( xw_1 + (1 - xy)w_2 = 1 \) for some related unit-regular \( w_1, w_2 \in R \). We easily check that \( x + (1 - xy)w_2s \in R \) is related unit for some \( s \in R \). Hence \( y + t(1 - xy) = w \), i.e., a related unit of \( R \). Consequently, we show that \( x = xyx = xwx \), as desired.

(1) \( \iff \) (3). Clear from the symmetry of related comparability. \( \square \)

Recall that a module \( M \) is quasi-injective if any homomorphism of a submodule of \( M \) into \( M \) extends to an endomorphism of \( M \). Now, we investigate rings which are quasi-injective as right modules. These extend the corresponding results in [3].

**Lemma 5.** Let \( R \) be quasi-injective as a right \( R \)-module. Given \( a, b \in R \) with \( aR + bR = R \), there exists some \( t \in R \) such that \( a + bt \) is a semi-related unit.

**Proof.** Given \( a, b \in R \) with \( aR + bR = R \), then \( \overline{a}(R/J(R)) + \overline{b}(R/J(R)) = R/J(R) \).

Since \( R \) is quasi-injective as a right \( R \)-module, by virtue of [9, Theorem 1], \( R/J(R) \) is a regular, right self-injective ring. Hence \( R \) is an exchange ring satisfying related comparability. According to Theorem 2, we can find a \( y \in R \) such that \( \overline{a} + \overline{b} \overline{y} = \overline{w} \) is a related unit of \( R/J(R) \). Therefore \( a + by = w + r \) for some \( r \in J(R) \). Clearly, \( w + r \) is a semi-related unit of \( R \), as desired. \( \square \)

**Theorem 6.** Let \( R \) be quasi-injective as a right \( R \)-module. Then the following hold:

1. Given \( a, b \in R \) with \( r \cdot \text{ann}(a) = r \cdot \text{ann}(b) \), there exists a semi-related unit \( w \in R \) such that \( a = wb \).

2. Given \( a, b \in R \) with \( l \cdot \text{ann}(a) = l \cdot \text{ann}(b) \), there exists a semi-related unit \( w \in R \) such that \( a = bw \).

**Proof.** (1) Given \( a, b \in R \) with \( r \cdot \text{ann}(a) = r \cdot \text{ann}(b) \). Since \( R \) is quasi-injective as a right \( R \)-module, by [3, Lemma 3.2], we have \( Ra = Rb \). Assume that \( a = sb, b = ta \) for some \( s, t \in R \). Then \( b = tsb \). Consequently, there exists some \( y \in R \) such that \( t + (1 - ts)y \) is a semi-related unit of \( R \) by Lemma 5. Using [5, Lemma 4], we have some \( z \in R \) such that \( s + z(1 - ts) = w \) is a semi-related unit of \( R \). Therefore, we claim that \( a = sb = (s + z(1 - ts))b = wb, \) as desired.

(2) Given \( a, b \in R \) with \( l \cdot \text{ann}(a) = l \cdot \text{ann}(b) \). Similarly to the consideration above, we have \( aR = bR \). Assume that \( a = bs, b = at \) for some \( s, t \in R \). Then \( b = bst \). From \( st + (1 - st) = 1 \), we can find a \( y \in R \) such that \( s + (1 - st)y = w \) is a semi-related unit of \( R \). Therefore \( a = bs = b(s + (1 - st)y) = bw \), whence the result. \( \square \)

**Corollary 7.** Let \( R \) be quasi-injective as a left \( R \)-module. Then the following hold:

1. Given \( a, b \in R \) with \( r \cdot \text{ann}(a) = r \cdot \text{ann}(b) \), there exists a semi-related unit \( w \in R \) such that \( a = wb \).

2. Given \( a, b \in R \) with \( l \cdot \text{ann}(a) = l \cdot \text{ann}(b) \), there exists a semi-related unit \( w \in R \) such that \( a = bw \).

**Proof.** Applying Theorem 6 to the opposite ring \( R^{\text{op}} \) of \( R \), we complete the proof. \( \square \)

**Theorem 8.** Let \( R \) be a ring which is quasi-injective as a right \( R \)-module. Then the following hold:
we claim that a semi-related unit. 

Moreover, we see that $R/J(R)$ satisfies related comparability. In view of Theorem 2, there exists $t \in R$ such that $a + tb = w + k$ with $w$ is a semi-related unit of $R$. Thus, there is some $k \in J(R)$ such that $a + tb = w + k$. Clearly, $w + k$ is also a semi-related unit. Thus, we claim that $a + tb$ is a semi-related unit of $R$.

(2) Given $a, b \in R$ with $1 \cdot \text{ann}(a) \cap 1 \cdot \text{ann}(b) = 0$, analogously to [3, Proposition 3.4], we claim that $a \cdot b + bR = R$. Thus, $(R/J(R)) \cdot a + (R/J(R)) \cdot b = R/J(R)$. Similarly to the consideration above, we show that $R/J(R)$ satisfies related comparability. In view of Theorem 2, there exists $t \in R$ such that $a + tb = w + k$ with $w$ is a semi-related unit and $k \in J(R)$. Since $w + k$ is also a semi-related unit, the result follows.

**Corollary 9.** Let $R$ be a ring which is quasi-injective as a left $R$-module. Then the following hold:

1. Given $a, b \in R$ with $r \cdot \text{ann}(a) \cap r \cdot \text{ann}(b) = 0$, there exists $t \in R$ such that $a + tb$ is a semi-related unit.
2. Given $a, b \in R$ with $1 \cdot \text{ann}(a) \cap 1 \cdot \text{ann}(b) = 0$, there exists $t \in R$ such that $a + bt$ is a semi-related unit.

**Proof.** Applying Theorem 8 to the opposite ring $R^{\text{op}}$ of $R$, we easily obtain the result.

Since every regular, right (left) self-injective ring is a quasi-injective ring with trivial Jacobson. As an immediate consequence of Theorem 6, Corollary 7, Theorem 8, and Corollary 9, we now derive the following.

**Corollary 10.** Let $R$ be a regular, right (left) self-injective ring. Then the following hold:

1. Given $a, b \in R$ with $r \cdot \text{ann}(a) = r \cdot \text{ann}(b)$, there exists a related unit $w \in R$ such that $a = wb$.
2. Given $a, b \in R$ with $1 \cdot \text{ann}(a) = 1 \cdot \text{ann}(b)$, there exists a related unit $w \in R$ such that $a = bw$.
3. Given $a, b \in R$ with $r \cdot \text{ann}(a) \cap r \cdot \text{ann}(b) = 0$, there exists $t \in R$ such that $a + tb$ is a related unit.
4. Given $a, b \in R$ with $1 \cdot \text{ann}(a) \cap 1 \cdot \text{ann}(b) = 0$, there exists $t \in R$ such that $a + bt$ is a related unit.

**References**


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