EXOTIC STRUCTURES ON QUOTIENT SPACES OF S³-ACTIONS

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(Received May 4, 1994)

ABSTRACT. A correct version of some results by A. Rigas regarding S³ actions on S⁷ × S³ and on
the symplectic group Sp₂ with quotients exotic seven-spheres is presented

KEY WORDS AND PHRASES: Exotic spheres, principal bundles, group actions

1991 AMS SUBJECT CLASSIFICATION CODES: 57R55, 57S25

1. INTRODUCTION

The present note is a result of our interest in finding exotic structures on 7-dimensional manifolds (cf
Guest and Micha [3], Astey, Micha and Pastor [1]) and its purpose is to correct some mistakes that occur
in a paper by A. Rigas [6]. Our contribution is simply to provide the correct statement and a different
proof of the key corollary that appears on page 76 of Rigas [6], but we take the opportunity to restate
several results of the paper which refer to the existence of free S³ actions on S⁷ × S³ and on the
symplectic group Sp₂ with quotients exotic seven-spheres, which also appear incorrectly stated in that
paper

2. MAIN RESULTS

We begin by recalling some definitions and notation of Rigas [6]. Principal S³ bundles over S⁴
are classified by π₁S³ which is naturally isomorphic to the group of integers Z. Let Pₙ denote the
total space of the bundle corresponding to the integer n. Similarly, the principal S³ bundles over S⁷
are classified by π₀S³. We shall denote by Eₙ the total space of the bundle corresponding to
i ∈ π₀S³ ≅ Z₁₂. The isomorphism here is such that E₁ ≅ Sp₂. Let ⁹Pₙ denote the pull-back of Pₙ
under the Hopf map S⁷ → S⁴. Then, as a principal S³ bundle, ⁹Pₙ is classified by the composition

\[ S⁷ \xrightarrow{h} S⁴ \xrightarrow{fₙ} S³ \rightarrow BS³ \]

where fₙ denotes the map of degree n, and the rightmost arrow is the inclusion of the bottom cell.

THEOREM. The bundles ⁹Pₙ and Eₙ(n−1)/2 are isomorphic as principal S³ bundles over S⁷.

This theorem is the correct version of the corollary on page 76 of Rigas [6]. The mistake leading to
the incorrect statement in Rigas [6] occurs in the calculation of the map fₙ o h, where the author fails to
iterate correctly a formula of Hilton [4]. An alternative proof using a different bundle decomposition is presented in §3 below.

It follows from the theorem that

(a) \( \vec{P}_n \) and the trivial bundle \( S^7 \times S^3 \) are isomorphic only if \( n \equiv 0, 1, 9 \) or 16 mod 24

(b) \( \vec{P}_n \) and the canonical bundle \( S^2_p \to S^7 \) are isomorphic only if \( n \equiv 2 \) or 23 mod 24

In particular, \( \vec{P}_{14} \) is not a trivial bundle. This renders §4 of Rigas [6] invalid. The theorem also allows us to rectify the statements of two important results of Rigas [6] as follows.

**COROLLARY.** There exist free actions of \( S^3 \) on \( S^7 \times S^3 \) with quotient the exotic seven-spheres of Eells-Kuiper invariants 16, 40 and 48.

**COROLLARY.** There exist free actions of \( S^3 \) on \( S^2_p \) with quotient the exotic seven-spheres of Eells-Kuiper invariants 2, 26, 34 and 42.

### 3. PROOF OF THE THEOREM

As is shown in Rigas [6], \( S^7 \) can be decomposed into two solid tori \( U \cong S^4 \times D^3 \) and \( V \cong D^4 \times S^1 \) such that the restriction of the bundle \( \vec{P}_n \) to each torus is trivial. Moreover, the transition map

\[ \lambda_{U/V} : S^3 \times S^3 \to S^3 \]

is given by

\[ \lambda_{U/V}(x, y) = x^{n-1}(yz^{-1})^{n-1} y^{n-1}, \]

where the group structure of unit quaternions is understood on \( S^3 \). Since the commutator \( xyz^{-1}y^{-1} \) generates \( \pi_6 S^3 \) (Hilton and Roitberg [5]) and since \( \lambda \) factors through \( S^6 \), the theorem is a consequence of the following result.

**PROPOSITION.** The map \( \lambda : S^3 \times S^3 \to S^3 \) given by \( \lambda(x, y) = x^{n-1}(yz^{-1})^{n-1} y^{n-1} \) is homotopic to \( (xyz^{-1}y^{-1})^{n(n-1)/2} \).

We first prove the following lemma.

**LEMMA.** The maps \( x^k/y^kz^{-k}y^{-l} \) and \( (xyz^{-1}y^{-1})^{kl} \) are homotopic.

**PROOF.** Consider the following commutative diagram:

\[
\begin{array}{ccc}
S^3 \times S^3 & \xrightarrow{\alpha} & S^3 \\
p \downarrow & & \downarrow \omega \\
S^6 & \to & S^6 \\
\end{array}
\]

where \( \alpha(x, y) = (x^k, y^k) \), \( \beta(x, y) = x^ky^{-1}y^{-1} \), \( \gamma(x) = x^{kl} \), \( p \) is the projection that collapses the 3-skeleton, \( f_{kl} \) is a map of degree \( kl \), and \( \omega \) is the generator of \( \pi_6 S^3 \). But since \( S^3 \) is an H-space, homotopy compositions are biadditive (Whitehead [7], p. 479), so \( \omega \circ f_{kl} \simeq \gamma \circ \omega \). Therefore,

\[ x^ky^lz^k \circ f_{kl} = \beta \circ \alpha \simeq \gamma \circ \beta = (xyz^{-1}y^{-1})^{kl} \]

We now prove the proposition by induction on \( n \). Let \( c = xyz^{-1}y^{-1} \). If we take \( k = 1 \) and \( l = -1 \) in the lemma we obtain \( xy^{-1}z^{-1}y \simeq c^{-1} = yxy^{-1}x^{-1} \). Hence,

\[ c^{-1}ycy^{-1} = (xyz^{-1}x^{-1})ycy^{-1} = y(xy^{-1}x^{-1}y)cy^{-1} \simeq yc^{-1}cy^{-1} = 1, \]

that is, \( cy^{-1} = y^{-1}c \).

Assume now that \( x^n(yx^{-1})y^n = c^{k(n)} \). Clearly, \( k(1) = 1 \). But now...
\[ x^n(yz^{-1})^n y^{-n} = x^n yz^{-1} (yz^{-1})^{n-1} y^{-n} \]
\[ = (x^n yz^{-n} y^{-1}) yz^{-1} (yz^{-1})^{n-1} y^{-n} \]
\[ = c^n y(x^{-1})^{n-1} y^{-1} (y^{-1})^{n-1} y^{-1} \]
\[ = c^n y c^{(n-1)} y^{-1} \]
\[ = c^{n+k(n-1)}. \]

Therefore, \( k(n) = n + k(n - 1) \), that is, \( k(n) = n(n + 1)/2 \) This proves the proposition.

REFERENCES


Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

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