ON k-IDEALS OF SEMIRINGS

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ABSTRACT. Certain types of ring congruences on an additive inverse semiring are characterized with the help of full $k$-ideals. It is also shown that the set of all full $k$-ideals of an additively inverse semiring in which addition is commutative forms a complete lattice which is also modular.

KEY WORDS AND PHRASES. Semiring, inverse semiring, $k$-ideals and ring congruence.

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1. PRELIMINARIES. A semiring is a system consisting of a non-empty set $S$ together with two binary operations on $S$ called addition and multiplication (denoted in the usual manner) such that

(i) $S$ together with addition is a semigroup;

(ii) $S$ together with multiplication is a semigroup; and

(iii) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.

A semiring $S$ is said to be additively commutative if $a + b = b + a$ for all $a, b \in S$. A left (right) ideal of a semiring $S$ is non-empty subset $I$ of $S$ such that

i) $a + b \in I$ for all $a, b \in I$; and

ii) $ra \in I$ (ar $\in I$) for all $r \in S$ and $a \in I$.

An ideal of a semiring $S$ is a non-empty subset $I$ of $S$ such that $I$ is both a left and right ideal of $S$.

Henriksen [1] defined a more restricted class of ideals in a semiring, which he called $k$-ideals.

A left $k$-ideal $I$ of a semiring $S$ is a left ideal such that if $a \in I$ and $x \in S$ and if either $a + x \in I$ or $x + a \in I$, then $x \in I$.

A right $k$-ideal of a semiring is defined dually. A non-empty subset $I$ of a semiring $S$ is called a $k$-ideal if it is both a left $k$-ideal and a right $k$-ideal.

A semiring $S$ is said to be additively regular if for each $a \in S$, there exists an element $b \in S$ such that $a = a + b + a$. If in addition, the element $b$ is unique and satisfies $b = b + a + b$, then $S$ is
called an **additively inverse semiring**. In an additively inverse semiring the unique inverse $b$ of an element $a$ is usually denoted by $a'$. Karvellas [2] proved the following result:

Let $S$ be an additively inverse semiring. Then

i) $x = (x')', (x + y)' = x' + y', (xy)' = x'y = y'x$ and $xy = x'y'$ for all $x, y \in S$.

ii) $E^+ = \{x \in S : x + x = x\}$ is an additively commutative semilattice and an ideal of $S$.

2. **FULL $k$-IDEALS.** In this section $S$ denotes an additively inverse semiring in which addition is commutative and $E^+$ denotes the set of all additive idempotents of $S$.

A left $k$-ideal $A$ of $S$ is said to be full if $E^+ \subseteq A$. A right $k$-ideal of $S$ is defined dually.

A non-empty subset $I$ of $S$ is called a full $k$-ideal if it is both left and a right full $k$-ideal.

**EXAMPLE 1.** In a ring every ring ideal is a full $k$-ideal.

**EXAMPLE 2.** In a distributive lattice with more than two elements, a proper ideal is a $k$-ideal but not a full $k$-ideal.

**EXAMPLE 3.** $Z \times Z^p = \{(a, b) : a, b$ are integers and $b > 0\}$. Define

$$(a, b) + (c, d) = (a + c, \text{l.c.m of } b, d) \text{ and } (a, b)(c, d) = (ac, h.c.f. of } b, d).$$

Then $Z \times Z^p$ becomes an additively inverse semiring in which addition is commutative.

Let $A = \{(a, b) \in Z \times Z^p : a = 0, b \in Z^p\}$. Then $A$ is a full $k$-ideal of $Z \times Z^p$.

**LEMMA 2.1.** Every $k$-ideal of $S$ is an additively inverse subsemiring of $S$.

**PROOF.** Let $I$ be a $k$-ideal of $S$. Clearly $I$ is a subsemiring of $S$. Let $a \in I$. Then

$$a + (a' + a) \in I.$$

Since $I$ is a $k$-ideal, it follows $a' + a \in I$. Hence the lemma.

**LEMMA 2.2.** Let $A$ be an ideal of $S$. Then

$$A = \{a \in S : a + x \in A \text{ for some } x \in A\}$$

is a $k$-ideal of $S$.

**PROOF.** Let $a, b \in A$. The $a + x, b + y \in A$ for some $x, y \in A$. Now

$$a + x + b + y = (a + b) + (x + y) \in A.$$

As $x + y \in A, a + b \in A$. Next let $r \in S, ra + rx = r(a + x) \in A$.

As $rx \in A, ra \in A$. Similarly, $ar \in A$. As a result $A$ is an ideal of $S$. Next, let $c$ and $c + d \in A$.

Then there exists $x$ and $y$ in $A$ such that $c + x \in A$ and $c + d + y \in A$.

Now

$$d + (c + x + y) = (c + d + y) + x \in A \text{ and } c + x + y \in A.$$

Hence $d \in A$ and $A$ is a $k$-ideal of $S$. Since $a + a' \in A$ for all $a \in A$, it follows that $A \subseteq A$.

**COROLLARY.** Let $A$ be an ideal of $S$. Then $A = A$ iff $A$ is a $k$-ideal.

**LEMMA 2.3.** Let $A$ and $B$ be two full $k$-ideals of $S$, then $A + B$ is a full $k$-ideal of $S$ such that $A \subseteq A + B$ and $B \subseteq A + B$.

**PROOF.** It can be shown that $A + B$ is an ideal of $S$. Then from Lemma 2.2, we find $A + B$ is a $k$-ideal and $A + B \subseteq A + B$. Now $E^+ \subseteq A, B$. Hence $E^+ \subseteq A + B \subseteq A + B$. This implies that $A + B$ is a full $k$-ideal. Let $a \in A$. Then

$$a = a + a' + a = a + (a' + a) \in A + B$$

as $a' + a \in E^+ \subseteq B$.

Hence $A \subseteq A + B$ and similarly $B \subseteq A + B$. 
THEOREM 2.4. If $I(S)$ denotes the set of all full $k$-ideals of $S$, then $I(S)$ is a complete lattice which is also modular.

PROOF. We first note that $I(S)$ is a partially ordered set with respect to usual set inclusion. Let $A, B \in I(S)$. Then $A \cap B \in I(S)$ and from Lemma 2.3, $A + B \in I(S)$. Define $A \wedge B = A \cap B$ and $A \vee B = A + B$. Let $C \in I(S)$ such that $A, B \subseteq C$. Then $A + B \subseteq C$ and $\overline{A + B} \subseteq \overline{C}$. But $\overline{C} = C$. Hence $\overline{A + B} \subseteq C$. As a result $\overline{A + B}$ is the l.u.b. of $A, B$. Thus we find that $I(S)$ is a lattice. Now $E^+$ is an ideal of $S$. Hence $\overline{E^+} \in I(S)$ and also $S \in I(S)$; consequently $I(S)$ is a complete lattice. Next suppose that $A, B, C \in I(S)$ such that

$$A \wedge B = A \wedge C \quad \text{and} \quad A \vee B = A \vee C \quad \text{and} \quad B \subseteq C.$$ 

Let $x \in C$. Then $x \in A \cap B = A \cap C$. Hence there exists $a + b \in A + B$ such that $x + a + b = a_1 + b_1$ for some $a_1 \in A, b_1 \in B$. Then

$$x + a + a' + b = a_1 + b_1 + a'.$$

Now $x \in C, a + a' \subseteq C$ and $b \in B \subseteq C$. Hence $a_1 + b_1 + a' \in C$. Consequently, $a_1 + a' \in C \cap A = C \cap B$. Hence $a_1 + a' \in B$. So from $x + a + b = a_1 + b_1$ we find that $x + a + a' + b = a_1 + a' + b \in B$. But $(a + a') + b \in B$ and $B$ is a $k$-ideal. Hence $x \in B$ and $B = C$. This proves that $I(S)$ is a modular lattice.

3. RING CONGRUENCES.

A congruence $\rho$ on a semiring $S$ is called a ring congruence if the quotient semiring $S/\rho$ is a ring.

In this section we assume $S$ is an additively inverse semiring in which addition is commutative. We want to characterize those ring congruences on $S$ such that $- (ap) = a' \rho$ where $a'$ denotes the inverse of $a$ in $S$ and $- (ap)$ denotes the additive inverse of $ap$ in the ring $S/\rho$.

THEOREM 3.1. Let $A$ be a full $k$-ideal of $S$. Then the relation

$$\rho_A = \{(a, b) \in S \times S : a + b' \in A\}$$

is a ring congruence on $S$ such that $- (a \rho_A) = a' \rho_A$.

PROOF. Since $a + a' \in E^+ \subseteq A$ for all $a \in S$, it follows that $\rho_A$ is reflexive. Let $a + b' \in A$. Now from Lemma 2.1, we find that $(a + b')' \in A$. Then $b + a' = (b')' + a' = (a + b')' \in A$. Hence $\rho_A$ is symmetric. Let $a + b' \in A$ and $b + c' \in A$. Then $a + b + b' + c' \in A$. Also $b + b' \in E^+ \subseteq A$. Since $A$ is a $k$-ideal, we find that $a + c' \in A$. Hence $\rho_A$ is an equivalence relation. Let $(a, b) \in \rho_A$ and $c \in S$. Then $a + b' \in A$. Since

$$(c + a) + (c + b')' = c + a + b' + c' = (a + b') + (c + c') \in A, \quad ca + (cb)' = ca + cb' = c(a + b') \in A,$$

it follows that $\rho_A$ is a congruence on $S$. So we obtain the quotient semiring where addition and multiplication are defined by

$$a \rho_A + b \rho_A = (a + b) \rho_A \quad \text{and} \quad (a \rho_A)(b \rho_A) = (ab) \rho_A.$$ 

Now

$$a \rho_A + b \rho_A = (a + b) \rho_A = (b + a) \rho_A = b \rho_A + a \rho_A.$$ 

Let $e \in E^+$ and $a \in S$. Now $(e + a) + a' = e + (a + a') \in E^+$. We find that $(e + a) \rho_A = a \rho_A$. Then $e \rho_A + a \rho_A = a \rho_A$. 

Also

\[ a\rho_A + a'\rho_A = (a + a')\rho_A = e\rho_A. \]

Hence \( e\rho_A \) is the zero element and \( a'\rho_A \) is the negative element of \( a\rho_A \) in the ring \( S/\rho_A \).

**THEOREM 3.2.** Let \( \rho \) be a congruence on \( S \) such that \( S/\rho \) is a ring and \( -(a\rho) = a'\rho \). Then there exists a full \( k \)-ideal \( A \) of \( S \) such that \( \rho_A = \rho \).

**PROOF.** Let \( A = \{ a \in S : (a, e) \in \rho \text{ for some } e \in E^+ \} \). Since \( \rho \) is reflexive, it follows that \( E^+ \subseteq A \). Then \( A \neq \phi \), since \( E^+ \neq \phi \). Let \( a, b \in A \). Then there exist \( e, f \in E^+ \) such that \((a, e) \in \rho \) and \((b, f) \in \rho \). Then \((a + b, e + f) \in \rho \). But \( e + f \in E^+ \). Hence \( a + b \in A \). Again for any \( r \in S \), \((ra, re) \in \rho \) and \((ar, er) \in \rho \). But \( re \) and \( er \in E^+ \). Hence \( A \) is an ideal of \( S \).

Let \( a + b \in A \) and \( b \in A \). Then there exist \( e, f, e' \in E^+ \) such that \((a + b, e) \in \rho \) and \((b, e') \in \rho \).

Hence \( f\rho = (a + b)\rho = a\rho + b\rho = a\rho + e\rho \). But \( f\rho \) and \( e\rho \) are additive idempotents in the ring \( S/\rho \). Hence \( e\rho = f\rho \) is the zero element of \( S/\rho \). As a result, \( a\rho \) is the zero element of \( S/\rho \). Then \( a\rho = e\rho \). This implies \( a \in A \). So we find that \( A \) is a full \( k \)-ideal of \( S \). Consider now the congruences \( \rho_A \) and \( \rho \). Let \((a, b) \in \rho \). Then \((a + b', b + b') \in \rho \). But \( b + b' \in E^+ \). Hence \( a + b' \in A \) and \((a, b) \in \rho_A \). Conversely suppose that \((a, b) \in \rho_A \). Then \( a + b' \in A \). Hence \((a + b', e) \in \rho \) for some \( e \in E^+ \). As a result, \( e\rho = a\rho + b'\rho = a\rho - b\rho \) holds in the ring \( S/\rho \). But \( e\rho \) is the zero element of \( S/\rho \). Consequently \( a\rho = b\rho \). This show that \((a, b) \in \rho \) and hence \( \rho_A = \rho \).

**REFERENCES**


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