REMARKS ON QUASILINEAR EVOLUTIONS EQUATIONS

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ABSTRACT. In this paper we study the existence result of classical solutions for the quasilinear equation
\[ u_{tt} - \Delta u - H \int_{\Omega} |\nabla u|^2 dx \Delta u_{tt} = f \]
with initial data \( u(0) = u_0, \ u_t(0) = u_1 \) and homogeneous boundary conditions.

KEY WORDS.- Partial differential equation, quasilinear evolution equation, boundary problem.

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1. INTRODUCTION: Let \( \Omega \) be an open and bounded set of \( \mathbb{R}^n \), with smooth boundary \( \Gamma \). Let's denote by \( Q \) the cylinder \( Q = \Omega \times [0,T] \) and by \( \Sigma \) its lateral boundary. Our notations and function spaces are standart and follows the same pattern as Lions's book [2].

Ebihara et al [1] was proved that there exist only one classical solution for a semilinear model, given by following initial-boundary value problem
\[
\begin{align*}
  u_{tt} - \Delta u - H \int_{\Omega} |\nabla u|^2 dx \Delta u_{tt} &= f & \text{in } Q \quad (1.1) \\
  u(0) &= u_0, \ u_t(0) &= u_1 & \text{in } \Omega \quad (1.2)
\end{align*}
\]
\[ u(x,t) = 0 \quad \text{in} \; \Sigma \quad (1.2) \]

when the following hypotheses hold:

(i) \( M(\lambda) \in C([0, +\infty) \) and there exist positive constants \( a, \rho \) such that the following inequality is valid:
\[ M(\lambda) \geq a\lambda^2 + \rho, \quad \forall \lambda \in (0, +\infty) \]

(ii) There exists a non-negative function \( K(\lambda) \) satisfying:
\[ |\frac{d}{d\lambda} M(\lambda)| \lambda \leq K(\lambda) \lambda \quad \forall \lambda \geq 0 \]

(iii) The initial data are such that:
\[ u_0, \; u_1 \in D(A^{(l+1)/2}), \; l \geq 1 \]
\[ f, \; \frac{d}{dt} f \in C(0, T; D(A^{1/2})), \; l \geq 1 \]

Where \( A = -\Delta \) and for \( D(A^\alpha) \) we are denoting the domain of the operator \( A^\alpha \). The main result of this paper is to prove the existence result of classical solutions for system (1.1)-(1.3) when

Hi. \( M \) is a continuous function such that: \( M(\lambda) \geq m_0 > 0 \)

H2. \( f \in C(0, T; D(A^{1/2})) \), \( l \geq 2 \) and \( u_0, \; u_1 \in D(A^{(l+1)/2}), \; l \geq 2 \)

2. THE MAIN RESULT: Let's denote by \( w_1, ..., w_m \) and by \( \lambda_1, ..., \lambda_m \) the \( m \) first orthonormal eigen functions and eigen values of the Laplacian respectively. Let's denote by \( V_m \) the finite dimensional vector space generated by the first \( m \) eigen functions and by \( P_m \) the projector operator on \( V_m \), that is:
\[ P_m v = \sum_{i=1}^{m} \left( \int_{\Omega} v(x) w_i(x) dx \right) w_i \]

It is easy to see that \( A^2 P_m = P_m A^2 \) in \( D(A^2) \). Moreover, we have that
\[ \int_{\Omega} |P_m w|^2 dx \leq \int_{\Omega} |w|^2 dx \quad (2.1) \]

Then the approximated problem is defined as follows.
\[ u_{tt}^{(m)} - \Delta u^{(m)} - M \int_{\Omega} |\nabla u^{(m)}|^2 dx \Delta u^{(m)} = f_m \quad (2.2) \]
\[ u^{(m)}(0) = u_0^m, \quad u_t^{(m)}(0) = u_1^m \text{ in } \Omega \]

where
\[ u^{(m)}(t) = \sum_{i=1}^{m} s_i^{(m)}(t) w_i, \quad u_0^m = P_m u_0, \quad u_1^m = P_m u_1 \]

Before to prove the main result of this paper we will show the following Lemmas:

**LEMMA 2.1** Let's suppose that \( u, \; u_t, \; u_{tt} \in C(0, T; L^2(\Omega)) \) and
\[ \int_{\Omega} |u_{tt}(x,t)|^2 dx \leq a + b \int_{\Omega} |u(x,t)|^2 dx \]
Then we have:
\[
\int_{\Omega} |u(x,t)|^2 dx \leq C e^{\frac{1}{2} \int_{0}^{t} \int_{\Omega} |\nabla u(x,\xi)|^2 d\xi dx} e^{\frac{1}{2} \int_{0}^{t} \int_{\Omega} |u(x,\xi)|^2 d\xi dx}
\]

**PROOF.** Since
\[
v(x,t) = \int_{0}^{t} u_t(x,\xi) d\xi + u(x,0) \quad \text{a.e. in } \Omega
\]
we have:
\[
|v(x,t)| \leq \sqrt{\int_{0}^{t} \int_{\Omega} |u_t(x,\xi)|^2 d\xi dx} + |u(x,0)|
\]
From where it follows
\[
\int_{\Omega} |v(x,t)|^2 dx \leq 2 \int_{0}^{t} \int_{\Omega} |u_t(x,\xi)|^2 d\xi dx + 2 \int_{\Omega} |u(x,0)|^2 dx
\]
Applying the relation above to \( u_t \) we have:
\[
\int_{\Omega} |u_t(x,t)|^2 dx \leq 2t \int_{0}^{t} \int_{\Omega} |u_{tt}(x,\xi)|^2 d\xi dx + 2t \int_{\Omega} |u(x,0)|^2 dx
\]
From the two last inequalities we conclude:
\[
\int_{\Omega} |v(x,t)|^2 dx \leq 4t |u(x,0)|^2 dx + 2 \int_{\Omega} |u(x,0)|^2 dx
\]
Finally, from the hypotheses, the last inequality and Gronwall's inequality the result of Lemma 2.1 follows.

**LEMMA 2.2.** Let suppose that \( w \in C([0,T];L^2(\Omega)) \), then we have that \( P_m w \to w \) strong in \( C([0,T];L^2(\Omega)) \).

**PROOF.** By the pointwise convergence of \( P_m w \) in \( t \), it's sufficient to show that \( P_m w \) is a Cauchy sequence in \( C([0,T];L^2(\Omega)) \). Let's take \( \varepsilon > 0 \), by the continuity of \( w \) we have that there exist \( \delta > 0 \) such that
\[
|t - s| < \delta \Rightarrow \int_{\Omega} |u(x,t) - u(x,s)|^2 dx < \frac{\varepsilon}{3} \quad (2.3)
\]
By the compactness of \([0,T]\), there exist \( s_1, s_2, \ldots, s_N \) satisfying
\[
[0,T] \subset \bigcup_{i=1}^{N} ]s_i - \delta, s_i + \delta[
\]
and from the pointwise convergence of \( P_m w \) we conclude that there exists a positive number \( N \) such that
\[
\int_{\Omega} |P_m w(x,s_i) - P_m w(x,s_i)|^2 dx < \frac{\varepsilon}{3} \quad \forall m, n \geq N, i = 1, \ldots, N \quad (2.4)
\]
Finally by (2.1), (2.3), (2.4) and the following inequality
\[
\left( \int_{\Omega} |P_m w(x,t) - P_m w(x,t)|^2 dx \right)^{1/2} \leq \left( \int_{\Omega} |P_m w(x,t) - P_m w(x,s_i)|^2 dx \right)^{1/2} + \left( \int_{\Omega} |P_m w(x,s_i) - P_m w(x,s_i)|^2 dx \right)^{1/2} + \left( \int_{\Omega} |P_m w(x,s_i) - P_m w(x,t)|^2 dx \right)^{1/2}
\]
the result of Lemma 2.2 follows.

**THEOREM 2.3.** Let's suppose that \( H_1 \) and \( H_2 \) are valid. Then there exists
(1.1), (1.2) and (1.3). Remains to show that \( u \) is a classical solution. Let's note that \( u_{(m)}^{(\mu)} \) belongs to \( C^2(\Omega;\mathcal{A}^{(l+1/2)}) \) for all \( m \in \mathbb{N} \), then in order to prove that \( u \in C^2(\Omega;\mathcal{C}(\Omega)) \), we will show that \( (u_{(m)}^{(\mu)})_{m \in \mathbb{N}} \) is a Cauchy's sequence in \( L^\infty(\Omega;\mathcal{A}^{(l+1/2)}) \), for all \( l \geq 2 \). In fact let \( \mu \in \mathbb{N} \), then

\[
u_{(m)}^{(\mu)} - \Delta \nu_{(m)}^{(\mu)} - \mathcal{M} \int_\Omega |\nabla u_{(m)}^{(\mu)}|^2 dx \Delta u_{(m)}^{(\mu)} = \mu f
\]

From (2.2) and the above equation we have:

\[
C \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} - \Delta \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} - \mathcal{M} \int_\Omega |\nabla u_{(m)}^{(\mu)}|^2 dx \Delta \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} = \mathcal{M} f
\]

where

\[
\mathcal{M} f = \left( \mathcal{M} \int_\Omega |\nabla u_{(m)}^{(\mu)}|^2 dx - \mathcal{M} \int_\Omega |\nabla u_{(m)}^{(\mu)}|^2 dx \Delta u_{(m)}^{(\mu)} + \mu f - \mu f \right)
\]

Multiplying the system above by \( \mathcal{M} \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} \) and integrating in \( \Omega \) we have

\[
\mathcal{M} \int_\Omega \left| \mathcal{M} \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} \right|^2 dx \leq \mathcal{M} \int_\Omega \left| \mathcal{M} \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} \right|^2 dx + \mathcal{M} \int_\Omega \left| \mathcal{M} \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} \right|^2 dx
\]

From which it follows that:

\[
\mathcal{M} \int_\Omega \left| \mathcal{M} \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} \right|^2 dx \leq \mathcal{M} \int_\Omega \left| \mathcal{M} \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} \right|^2 dx + \mathcal{M} \int_\Omega \left| \mathcal{M} \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} \right|^2 dx
\]

From Lemma (3.1) and the last inequality we have

\[
\mathcal{M} \int_\Omega \left| \mathcal{M} \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} \right|^2 dx \leq \mathcal{M} \int_\Omega \left| \mathcal{M} \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} \right|^2 dx + \mathcal{M} \int_\Omega \left| \mathcal{M} \frac{u_{(m)}^{(\mu)}}{u_{(m)}^{(\mu)}} \right|^2 dx
\]

Finally from Lemma 2.2 and since \( u_0, u_1 \in \mathcal{A}^{(l+1/2)} \) we have that

\[
A^{l/2} \mathcal{M} f + 0 \quad \text{as} \quad m, \mu \to + \infty \quad \text{strongly in} \quad C(\Omega,\mathcal{L}^2(\Omega))
\]

Then we have that \( (u_{(m)}^{(\mu)}) \) a Cauchy sequence in \( L^\infty(\Omega;\mathcal{A}^{(l+1/2)}) \) and the proof is now complete.

**REMARK 2.4. UNIQUENESS:** If \( H \) is locally Lipschitz, then we have uniqueness. In fact, let \( u \) and \( v \) be two solutions, putting \( w = u - v \) we have

\[
w_{(m)} - \Delta w - \mathcal{M} \int_\Omega |\nabla u_{(m)}|^2 dx \Delta w_{(m)} = \mathcal{M} \int_\Omega |\nabla u_{(m)}|^2 dx - \mathcal{M} \int_\Omega |\nabla u_{(m)}|^2 dx \Delta w_{(m)}
\]

Multiplying by \( \Delta w_{(m)} \) applying HI and the Lipschitz condition on \( H \) we have that there exists a positive constant \( c_1 \) such that:

\[
\mathcal{M} \int_\Omega |\Delta w_{(m)}|^2 dx \leq \mathcal{M} \int_\Omega |\Delta w_{(m)}|^2 dx + c_1 \mathcal{M} \int_\Omega |\Delta w_{(m)}|^2 dx + c_1 \mathcal{M} \int_\Omega |\Delta w_{(m)}|^2 dx
\]
only one classic solution of system \((1.1), (1.2)\) and \((1.3)\)

**Proof.** Since \(DK^{(1+1)/2}\) \(\subset H^{1+1}(\Omega) \subset C^0(\Omega)\) if \(1+1 > \frac{n}{2} + k\), it's sufficient to show that there exists a solution of system \((1.1), (1.2)\) and \((1.3)\) satisfying \(v \in C^0(0,T;DK^{(1+1)/2})\). In order to prove it let's multiply \((2.2)\) by \(A_t^{(m)}u_{tt}\) and integrating in \(\Omega\) we have:

\[
\int_\Omega |A^2 u^{(m)}| dx + \int_\Omega K \int \nabla u^{(m)} \nabla u^{(m)} dx \int A^2 u^{(m)} dx = \int_\Omega A_t u^{(m)} A_t^{(m)} dx + \int_\Omega f |A^2 u^{(m)} dx.
\]

By \(H1\) and \(H2\) the last equality becomes:

\[
m_\delta \int_\Omega |A^2 u^{(m)}|^2 dx \leq \int_\Omega |A^2 u^{(m)} A^2 u^{(m)}| dx + \int_\Omega |A^2 f A^2 u^{(m)}| dx
\]

from where it follows that:

\[
\frac{1}{\lambda^2} \int_\Omega |A^2 u^{(m)}|^2 dx \leq \int_\Omega |A^2 f A^2 u^{(m)}|^2 dx + \int_\Omega |A^2 u^{(m)}|^2 dx.
\]

By Lemma \(2.1\) and the above inequality we obtain:

\[
\frac{1}{\lambda^2} \int_\Omega |A^2 u^{(m)}(x,t)|^2 dx \leq \int_\Omega |A^2 f A^2 u^{(m)}|^2 dx + 2 \int_\Omega |A^2 u^{(m)}|^2 dx + 4 \int_\Omega |A^2 u^{(m)}|^2 dx Exp\left(\frac{\rho}{m_\delta} t^4\right).
\]

From \((2.5)\) and since:

\[
\int_\Omega |A^2 u^{(m)}(x,t)|^2 dx \leq 2 \int_\Omega |A^2 u^{(m)}(x,t)|^2 dx + 2 \int_\Omega |A^2 u^{(m)}|^2 dx
\]

we conclude that there exists a subsequence of \(\{u^{(m)}\}_{m \in \mathbb{N}}\) which we still denoting of the same way and a function \(u \in L^0(0,T;DK^{(1+1)/2})\), satisfying

\[
\begin{align*}
\liminf_{m \to \infty} u^{(m)}(x,t) &= u(x,t) \quad \text{weak star in } L^0(0,T;DK^{(1+1)/2}) \\
\liminf_{m \to \infty} u^{(m)}(x,t) &= u(x,t) \quad \text{weak star in } L^0(0,T;DK^{(1+1)/2}) \\
\liminf_{m \to \infty} u^{(m)}(x,t) &= u(x,t) \quad \text{weak star in } L^0(0,T;DK^{(1+1)/2})
\end{align*}
\]

From the last convergences and the Lions-Aubin's theorem (see Lions's [2], theorem 5.1, chap 1) we conclude in particular that:

\[
u^{(m)} \to u\quad \text{strongly in } C(0,T;H^0(\Omega)) \quad \text{as } m \to \infty
\]

By standard methods we can prove that \(u\) is a strong solution of system
from where it follows that there exists $c_2$ such that:

$$\int_{\Omega} |\Delta \omega|^2 dx \leq c_2 \int_{\Omega} |\omega|^2 dx$$

By Lemma 2.1, since $\omega(x,0) = \omega_t(x,0) = 0$, we obtain that $\Delta \omega = 0$, and from this it follows that $\omega = 0$, that is $u = v \circ$.

REFERENCES


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