A NOTE ON SOME SPACES $L_\gamma$ OF DISTRIBUTIONS WITH LAPLACE TRANSFORM

SALVADOR PÉREZ ESTEVA

Instituto de Matemáticas
Universidad Nacional Autónoma de México
México, D.F. 04510 México

(Received March 2, 1989)

ABSTRACT. In this paper we calculate the dual of the spaces of distributions $L_\gamma$ introduced in [1]. Then we prove that $L_\gamma$ is the dual of a subspace of $C^\infty(\mathbb{R})$.

KEY WORDS AND PHRASES. Convolution, Laplace Transform, Strict Inductive Limit.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. Primary 44A35, Secondary 44A10

1. INTRODUCTION

Let $\mathcal{D}'$ and $\mathcal{S}'$ be the classical Schwartz's spaces of distributions in $\mathbb{R}$ and denote by $L$ the Laplace transformation. In (Pérez-Esteva [1]) were introduced spaces $L^a_\gamma$ as follows: $L^a_\gamma$ is the subspace of $L^1_{\text{loc}}(\mathbb{R})$ of functions $f$ with $\text{supp } f \subseteq [a,\infty)$ and $e^{-\gamma f} \in L^2(\mathbb{R})$, where $e^{-\gamma x} = e^{-\gamma x}$. $L^a_\gamma$ is a Hilbert space with the inner product

$$
(f,g) = \int e^{-\gamma x} f(x) g(x) \, dx
$$

then we define $L^a_\gamma = \mathcal{D} L^a_\gamma$, where $\mathcal{D}$ is the distributional derivative of order $p$.

Since $\mathcal{D}^p: L^a_\gamma \rightarrow L^a_{p\gamma}$ is bijective, we can copy the Hilbert space structure of $L^a_{p\gamma}$ on $L^a_{p\gamma}$. We have the continuous inclusions

$$
L^a_{p\gamma} \subseteq L^b_{p\gamma}, \text{ for } a > b
$$

$$
L^a_{p\gamma} \subseteq L^a_{q\gamma}, \text{ if } p < q
$$

Hence for $p = \{0,1,\ldots\}$ the strict inductive limit

$$
L_\gamma = \text{ind lim}_{a \rightarrow -\infty} L^a_{p\gamma}
$$

makes sense. Then

$$
L_\gamma = \text{ind lim}_{p \rightarrow \infty} L_\gamma = \text{ind lim}_{p \rightarrow \infty} L^{-p}_\gamma
$$

is also well defined.

In [1] it was studied the spaces of distributions $g$ for which the convolution

$$
f \ast f \ast g: L_\gamma \rightarrow L_\gamma
$$

is continuous.
Here we describe the strong dual of $L_\gamma$, which turns out to be a subspace $S_\gamma$ of $C^\infty_0(\mathbb{R})$. Then we prove the reflexivity of $S_\gamma$, and conclude that $(S_\gamma)' = L_\gamma$, which is the main result of the paper. $\|\cdot\|_2$ will denote the norm of $L^2(\mathbb{R})$, $\gamma$ will be assumed to be a positive constant, and $N$ will be the set of nonnegative integers.

2. THE DUAL OF $L_\gamma$

**Definition 1.** Let $L$ be the space of all complex measurable functions $g$ in $\mathbb{R}$ such that $\chi_{[a,\infty)}e^{-\gamma g} \in L^2_\gamma$ for every $a \in \mathbb{R}$, where $\chi_{[a,\infty)}$ stands for the characteristic function of $[a,\infty)$. We provide $L_\gamma$ with the topology given by the seminorms

$$P_a(g) = \|\chi_{[a,\infty)}e^{-\gamma g}\|_2, \quad a \in \mathbb{R}.$$ 

Next we denote by $S_\gamma$ the subspace of $L_\gamma$ such that $D^ng \in L_\gamma$ for every $n \in N$. Define the topology of $S_\gamma$ by the system of seminorms

$$P_n(g) = \|\chi_{[a,\infty)}e^{-\gamma D^n g}\|_2, \quad a \in \mathbb{R}, \quad n \in N.$$ 

It is clear that $L_\gamma$ and $S_\gamma$ are Fréchet spaces and since $D^ng \in L^1_{loc}(\mathbb{R})$ for any $n \in N$ and $g \in S_\gamma$, we have that $S_\gamma \subset C^\infty_0(\mathbb{R})$.

**Lemma 1.** Let $\phi \in L_\gamma'$, then for every $p \in N$, there exists $g_p \in L_\gamma$ such that

$$\phi(D^pf) = \int e^{-2\gamma f} g_p \, dx, \quad f \in L^1_{p\gamma}.$$ 

The sequence $\{g_p\}_{p \in N}$ satisfies

$$g_{p+1} = Dg_p + 2\gamma g_p, \quad p \in N \quad (2.1)$$

Hence $\phi$ is determined by $g_0 \in S_\gamma$.

**Proof.** Fix $a \in \mathbb{R}$ and $p \in N$. Then $\phi \in (L^a_{p\gamma})'$, and there exists $g_{pa} \in L^a_{p\gamma}$ such that

$$(D^pf) = \int e^{-2\gamma f} g_{pa} \, dx, \quad D^pf \in L^a_{p\gamma}$$

If $a < b$, we have $L^a_{p\gamma} \subset L^b_{p\gamma}$, then

$$\phi(D^pf) = \int e^{-2\gamma f} g_{pb} \, dx = \int e^{-2\gamma f} \chi_{[b,\infty)}g_{pa} \, dx$$

for $D^pf \in L^b_{p\gamma}$, which shows that

$$g_{pb} = \chi_{[b,\infty)}g_{pa}$$

If $g_{pa}$ is the restriction of $g_{pa}$ to $[a,\infty)$, then $g_p = \cup_a g_{pa}$ is well defined, belongs to $L_\gamma$ and

$$\phi(D^pf) = \int e^{-2\gamma f} g_p \, dx, \quad D^pf \in L^a_{p\gamma}$$

Let $\varphi \in D$. Since $D^{p+1}\varphi \in L^{p+1}_{p\gamma} \cap L^1_{p\gamma}$, we have
where \(<,\cdot,\cdot>\) represents the duality between \(\mathcal{D}\) and \(\mathcal{D}'\). It follows that
\[
g_{p+1} = -Dg_p + 2\gamma g_p
\]
or
\[
e^{-2\gamma}g_{p+1} = -D(e^{-2\gamma}g_p)
\]
Hence, every \(g_p\) belongs to \(S_\gamma\).

**LEMMA 2.** Let \(g \in S_\gamma\) and \(H\) be the differential operator defined by
\[H = -D + 2\gamma I.\]
Then the functional
\[
\phi(D^p f) = \int_{\mathbb{R}} e^{-2\gamma} f H(p) g dx, \quad f \in L^{0}_{\mathcal{O}_Y}
\]
is well defined in \(L^{0}_{\mathcal{O}_Y}\) and is continuous.

**PROOF.** Let \(f \in L^a_{\mathcal{O}_Y}\) be such that \(f = Dh\) with \(h \in L^{0}_{\mathcal{O}_Y}\). There exists a sequence \(\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{D}\) converging to \(f\) in \(L^b_{\mathcal{O}_Y}\) if \(b < a\).

Let
\[
\varphi_n(x) = \int_{-\infty}^{x} f_n dy
\]
Then \(f \in L^b_{\mathcal{O}_Y}\), \(D(\varphi_n - h) = f_n - f\), and since the inclusion \(L^b_{\mathcal{O}_Y} \subset L^b_{\mathcal{O}_Y}\) is continuous, we have that \(\{\varphi_n\}_{n \in \mathbb{N}}\) converges to \(h\) in \(L^b_{\mathcal{O}_Y}\). If follows that
\[
\int_{\mathbb{R}} e^{-2\gamma} h H(g) dx = \lim_{n \to \infty} \int_{\mathbb{R}} e^{-2\gamma} n H(g) dx \quad (2.2)
\]
and
\[
\int_{\mathbb{R}} e^{-2\gamma} f g dx = \lim_{n \to \infty} \int_{\mathbb{R}} e^{-2\gamma} f_n g dx \quad (2.3)
\]
On the other hand
\[
\int_{-\infty}^{B} e^{-2\gamma} \varphi_n H(g) dx = \int_{-\infty}^{B} \varphi_n D(e^{-2\gamma}g) dx
\]
\[
= -\varphi_n(B)e^{-2\gamma(B)}g(B) + \int_{B}^{\infty} f_n e^{-2\gamma} g dx \quad (2.4)
\]
But we have the estimate
\[
|g(x)| \leq |g(b)| + e_{\gamma}(x) \|\chi_{[b,\infty)} e_{-\gamma}(Dg - \gamma g)\|_2 (x-b)^{1/2} \quad \text{for } x > b
\]
Hence
\[
\int_{\mathbb{R}} e^{-2\gamma} \varphi_n H(g) dx = \int_{\mathbb{R}} e^{-2\gamma} f_n g dx
\]
From (2.2) and (2.3) it follows that
\[
\int_{\mathbb{R}} e^{-2\gamma} f g dx = \int_{\mathbb{R}} e^{-2\gamma} h H(g) dx \quad (2.5)
\]
By induction we obtain
\[
\int e^{-2\gamma}f \, dx = \int e^{-2\gamma} h(P) \, dx
\]  
(2.6)
if \( f = D^p h \) and \( f, h \in L_{\infty}^{\gamma} \).

Finally, if \( D^p f = D^p h \) with \( f, h \in L_{\infty}^{\gamma} \) and \( q \geq p \), then \( f = D^{q-p} h \), hence by (2.6) we have
\[
\int e^{-2\gamma} f \, H(p)(g) \, dx = \int e^{-2\gamma} h \, H(q)(g) \, dx
\]
Thus \( \Phi \) is well defined and it is clearly continuous.

**THEOREM 1.** The strong dual of \( L_{\gamma} \) is \( S_{\gamma} \).

**PROOF.** By lemmas 1 and 2 we know that \( L_{\gamma}' = S_{\gamma} \). It remains to prove that the strong topology \( \pi(L_{\gamma}', L_{\gamma}) \) coincides with the topology \( \tau \) of \( S_{\gamma} \). First notice that \( \tau \) is defined by the system of seminorms
\[
q_p(a) = \| \chi_{[a, \infty)} e^{-\gamma} H(p)(g) \|_2, \quad a \in \mathbb{R}, \quad p \in \mathbb{N}
\]
Fix \( a \in \mathbb{R} \) and \( p \in \mathbb{N} \). Let \( V = \{ g \in S_{\gamma}: q_p(g) < 1 \} \). Denote by \( U \) the unit ball in \( L_{\infty}^{\gamma} \), then the set \( B = D^p U \) is bounded in \( L_{\infty}^{\gamma} \) and hence in \( L_{\gamma} \). If \( g \in B \) (the polar of \( B \)), then for every \( f \in U \) we have
\[
\int e^{-2\gamma} f \, H(p)(g) \, dx = |<D^p f, g>| < 1
\]
Thus
\[
\| e^{-\gamma} \chi_{(a, \infty)} H(p)(g) \|_2 < 1
\]
It follows that \( B \subseteq V \) and \( \tau \subseteq B(L_{\gamma}', L_{\gamma}) \). Now, let \( B \) be a bounded set in \( L_{\gamma} \). Then for some \( p \in \mathbb{N}, B \subseteq L_{\infty}^{p} \) and is bounded there (see Kucera, McKennon [2]). Hence \( B \subseteq D^p U \) for some \( \varepsilon > 0 \), where \( U \) is the unit ball in \( L_{\infty}^{\gamma} \). Let \( V = \{ g \in S_{\gamma}: q_p(g) < \varepsilon^{-1} \} \), then \( g \in V \) implies for \( f \in \varepsilon U \) that
\[
<D^p f, g> = \int e^{-2\gamma} f \, H(p)(g) \, dx \leq 1
\]
Then \( g \in B \), so we proved that \( V \subseteq B \). This completes the proof.

**COROLLARY 1.** \( L_{\gamma} \) is the strong dual of \( S_{\gamma} \).

**PROOF.** By (Kucera, McKennon [2], Theorem 4) we know that \( L_{\gamma} \) is reflexive. Hence the corollary follows from Theorem 1.

**REFERENCES**


Special Issue on
Time-Dependent Billiards

Call for Papers
This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Deadline</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors
Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob’evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru