A NOTE ON ANALYTIC MEASURES
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ABSTRACT: Let $G$ be a compact Abelian group with character group $X$. Let $S$ be a subset of $X$ such that, for some real-valued homomorphism $\psi$ on $X$, the set $S \cap \psi^{-1}([-\infty, \psi(\chi)])$ is finite for all $\chi$ in $X$. Suppose that $\mu$ is a measure in $M(G)$ such that $\hat{\mu}$ vanishes off of $S$, then $\mu$ is absolutely continuous with respect to the Haar measure on $G$.

KEY WORDS AND PHRASES. Analytic measures, absolutely continuous, Bochner’s Theorem.

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1. INTRODUCTION.
Let $G$ denote a compact Abelian group with character group $X$. Suppose that $\psi$ is a real-valued homomorphism on $X$, and let $\phi$ denote the adjoint homomorphism of $\psi$. Thus $\phi$ is the continuous homomorphism from $\mathbb{R}$ into $G$ such that the identity $\chi \circ \phi(r) = \exp(\psi(\chi) r)$ holds for all $r$ in $\mathbb{R}$, and all $\chi$ in $X$. We denote by $M(G)$ the linear space of all complex-valued regular Borel measures on $G$. In the terminology of de Leew and Glicksberg [1], a measure $\mu$ in $M(G)$ is called $\phi$-analytic if its Fourier transform $\hat{\mu}$ vanishes on $\{ \chi \in X : \psi(\chi) < 0 \}$.

Suppose that $S$ is a nonvoid subset of $X$. Let $M_S(G)$ denote the closed linear subspace of $M(G)$ consisting of the measures $\mu$ with $\hat{\mu}$ vanishing off of $S$. The set $S$ will be called a $B$-set (B for Bochner) if there is a nonzero homomorphism $\psi$ from $X$ into $\mathbb{R}$ such that the set $S \cap \psi^{-1}([-\infty, \psi(\chi)])$ is finite for all $\chi$ in $X$. The homomorphism $\psi$ may depend on $S$, and may not be unique. For example, a sector with opening less than $\pi$ in the lattice plane $\mathbb{Z} \times \mathbb{Z}$ is a $B$-set. The first orthant in $\mathbb{Z}^d$ (the weak direct product of countably many copies of $\mathbb{Z}$) is also a $B$-set. Once we have chosen a homomorphism $\psi$, we will refer to $S$ as a $B$-set with respect to the homomorphism $\psi$.

A theorem due to Bochner [2], on $T^2$, the two-dimensional torus, asserts that if $\mu \in M(T^2)$ is such that $\hat{\mu}$ vanishes off of a sector of opening less than $\pi$, then $\mu$ is absolutely continuous. (The expression "absolutely continuous" will always mean absolutely continuous with respect to the Haar measure on the group in consideration.) A generalization of this result is given in de Leew and Glicksberg [1], Theorem (3.4).

It is easy to construct $B$-sets in $\mathbb{Z} \times \mathbb{Z}$ that are contained in no sector with opening less than $\pi$. For example, consider the set $S = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : y \geq \log(1 + |x|)\}$. Using results from [1], we will show that the conclusion of Bochner’s theorem holds for $B$-sets. We have the following theorem.

(1.1) THEOREM. Let $S$ be a $B$-set in $X$. Suppose that $\mu$ is in $M_S(G)$, then $\mu$ is absolutely continuous.
Before proving the theorem we make a few observations. Suppose that $S$ is a B-set, with respect to some homomorphism $\psi$. Clearly, there is a character $\chi_0$ in $S$ such that $\psi(\chi) \leq \psi(\chi_0)$ for all $\chi$ in $S$. Note that any translate of $S$ by an element of $X$ is also a B-set with respect to the same homomorphism $\psi$. Hence by shifting $S$ by $-\chi_0$, if necessary, we may suppose that $\psi(\chi) \geq 0$ for all $\chi$ in $S$. In this case, given a measure $\mu$ in $M_\Sigma(G)$, we consider the measure $\tilde{\mu}$ which is in $M_\Sigma(G)$. The set $S - \chi_0$ is a B-set, with respect to the homomorphism $\psi$; and $\tilde{\mu}$ is absolutely continuous if and only if $\mu$ is.

If $\mu$ is in $M(G)$, we write $\mu_a$ and $\mu_s$ to denote its absolutely continuous part and its singular part respectively.

(1.2) Lemma. Let $S$ be a B-set in $X$. Suppose that $\mu$ is in $M_\Sigma(G)$, then $\mu_a$ and $\mu_s$ are in $M_\Sigma(G)$.

Proof. As we observed before the lemma, we may suppose that $\psi(S) \subseteq [0,\infty]$. Let $\phi$ denote the adjoint homomorphism of $\psi$, and let $\chi_1$ be an arbitrary character in $X \setminus S$, the complement of $S$ in $X$. We want to show that

\[ \hat{\mu_a}(\chi_1) = \hat{\mu_s}(\chi_1) = 0. \]

First, note that if $S$ is finite then $\mu = \mu_a$, and the lemma is obviously true. So suppose for the rest of the proof that $S$ is infinite. Let $\chi_2$ in $X$ be such that $\psi(\chi) < \psi(\chi_2)$. Let $A = \{ \chi \in X : \psi(\chi) < \psi(\chi_2) \} \cap \text{supp} \hat{\mu}$. The set $A$ is either void or finite. Define the measure $\sigma$ in $M(G)$ by

\[ \sigma = \mu - \sum_{\chi \in A} \hat{\mu}(\chi) \chi, \]

where the above sum is 0 if $A$ is empty. We have

\[ \hat{\sigma}(\chi) = \begin{cases} \hat{\mu}(\chi) & \text{if } \chi \notin A; \\ 0 & \text{if } \chi \in A. \end{cases} \]

Hence $\hat{\sigma}$ vanishes off of $\psi^{-1}([\psi(\chi_2), \infty]) \cap S$, which implies that $\sigma$ is $\phi$-analytic. It follows from [1], the Main Theorem, Proposition (2.3.2), and Theorem (5.1), that $\hat{\sigma}_a$ and $\hat{\sigma}_s$ vanish off of $\psi^{-1}([\psi(\chi_2), \infty]) \cap S$. Since $\mu_a = \sigma_a$, it follows that $\hat{\mu}_a$ vanishes off of $\psi^{-1}([\psi(\chi_2), \infty]) \cap S$. Therefore, $\hat{\mu}_a(\chi_1) = 0$, and the lemma follows.

Proof of Theorem (1.1). According to Lemma (1.2), it is enough to show that $\hat{\mu}_a(\chi) = 0$ for all $\chi$ in $S$. The proof is by contradiction. Assume that $\hat{\mu}_a(\chi_0) \neq 0$ for some $\chi_0$ in $S$. Let $\chi_1$ in $X$ be such that $\psi(\chi_1) > \psi(\chi_0)$. (Here also we are assuming that $S$ is infinite and $\psi(S) \subseteq [0,\infty]$.) Let $A = \{ \chi \in X : \psi(\chi) \leq \psi(\chi_1) \}$, and $\hat{\mu}_a(\chi_0) \neq 0$. Then $A$ is contained in $\psi^{-1}([\psi(\chi_1), \infty]) \cap S$, and so $A$ is finite and $\chi_0$ is in $A$. Define the measure $\nu$ in $M(G)$ by

\[ \nu = \nu = \sum_{\chi \in A} \hat{\mu}_a(\chi) \chi. \]

We have

\[ \hat{\nu}(\chi) = \begin{cases} \hat{\mu}_a(\chi) & \text{if } \chi \notin A; \\ 0 & \text{if } \chi \in A. \end{cases} \]

Thus $\hat{\nu}$ vanishes off of $\psi^{-1}([\psi(\chi_1), \infty]) \cap S$, and hence it is $\phi$-analytic. Applying Proposition (5.1), [1], we see that $\hat{\nu}_a$ and $\hat{\nu}_s$ vanish off of $\psi^{-1}([\psi(\chi_1), \infty]) \cap S$. Since $\nu_a = \mu_a$, it follows that $\hat{\mu}_a$ vanishes off of $\psi^{-1}([\psi(\chi_1), \infty]) \cap S$. This is plainly a contradiction since $\psi(\chi_0) < \psi(\chi_1)$, and by assumption $\hat{\mu}_a(\chi_0) \neq 0$.

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<td>June 1, 2009</td>
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