COMMUTATIVITY OF RINGS WITH CONSTRAINTS ON NILPOTENTS AND NONNILPOTENTS

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ABSTRACT. Let \( R \) be a ring (not necessarily with identity), \( N \) the set of nilpotents, and \( n > 1 \) a fixed integer. Suppose that (i) \( N \) is commutative; (ii) If \( x \notin N \) and \( y \notin N \), then \( x^n y = x y^n \); (iii) For \( a \in N \) and \( b \in R \), if \( n! [a,b] = 0 \), then \( [a,b] = 0 \), where \( [a,b] = ab - ba \) denotes the commutator. Then \( R \) is commutative. This theorem generalizes the "\( x^n = x \)" theorem of Jacobson. It is also shown that above theorem need not be true if any of the hypotheses is deleted, or if "\( n! \)" in (iii) is replaced by "\( n \)."

KEY WORDS AND PHRASES. Commutator, nilpotent, Vandermonde determinant.

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1. INTRODUCTION.

A well known theorem of Jacobson [2] states that a ring \( R \) satisfying the identity \( x^n = x \), \( n > 1 \) is fixed, is commutative. Such rings, of course, have no nonzero nilpotents. With this as motivation, we consider the commutativity of a ring satisfying the condition \( x^n y = x y^n \) for all \( x \in R \setminus N \), \( y \in R \setminus N \), \( n > 1 \) is fixed, and where \( N \) is assumed to be commutative. That such a ring \( R \) need not be commutative can be seen by taking

\[
R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mid 0,1 \in GF(2) \right\}, \quad n = 2.
\]
This naturally raises the following question: What extra conditions are needed to guarantee the commutativity of the ground ring $R$? Here we give one such extra condition involving commutators. As a corollary of our main theorem, we obtain Jacobson's Theorem (quoted above). We also give examples which show that all the hypotheses of our main theorem are essential.

2. MAIN RESULTS.

Our main result may be stated as follows:

**MAIN THEOREM.** Let $R$ be a ring (not necessarily with identity), $N$ the set of nilpotents, and $n > 1$ a fixed integer. Suppose that (i) $N$ is commutative; (ii) If $x \not\in N$ and $y \not\in N$, then $x^n y = xy^n$; (iii) For $a \in N$ and $b \in R$, if $n![a,b]=0$, then $[a,b]=0$. Then $R$ is commutative.

**PROOF.** Let $x \in R$, $x \not\in N$. Then $x^2 \not\in N$, and hence by (ii),

$$x^n x^2 = x^{2n} = x^{n+2},$$

which implies $x^{2n+1} = x^{n+2}$. Thus,

$$(x-x^n)(x-x^n)^{n+1} = (x-x^n)^{n+1} g(x) = 0,$$

and hence $x - x^n \in N$ for all $x \not\in N$. Since, trivially, this is also true if $x \in N$, therefore

$$x - x^n \in N \text{ for all } x \in R. \quad (2.1)$$

Next, we prove that

$$(n!)^\sigma [a,b] = 0 \text{ for some positive integer } \sigma, \quad (a \in N, \ b \in R). \quad (2.2)$$

Since $N$ is commutative, by (i), to prove (2.2) we may assume that $b \not\in N$. Let

$$u = a + b, \quad (a \in N, \ b \not\in N). \quad (2.3)$$

We now distinguish three cases.

**CASE 1.** $ku \in N$ for some $k \in \{1,\ldots,n\}$.

Since $a \in N$ and $N$ is commutative by (i), therefore $(ku)a = a(ku)$ and hence by (2.3), $k(a+b)a = ka(a+b)$. Thus, $k[a,b] = 0$ and hence $n![a,b] = 0$, which proves (2.2) in this case.

**CASE 2.** $b + ku \in N$ for some $k \in \{1,\ldots,n-1\}$.

Arguing as in Case 1, we see that $[b + ku, a] = 0$. Hence, $[b + k(a+b), a] = 0$, which implies $(k+1)[a,b] = 0, \ k + 1 < n$, and thus $n![a,b] = 0$. Again (2.2) is proved in this case.

**CASE 3.** $ku \not\in N$ for $k = 1,\ldots,n$ and $b + ku \not\in N$ for $k = 1,\ldots,n-1$. 

Recall that \( b \notin N \), [see (2.3)], and \( ku \notin N \) for \( k = 1, \ldots, n \). Hence by (ii),

\[
(ku)^n b = (ku)b^n \quad \text{for } k = 1, \ldots, n. \tag{2.4}
\]

Similarly, since \( b \notin N \) and \( b + ku \notin N \) for \( k = 1, \ldots, n-1 \), therefore by (ii) again,

\[
(b+ku)^n b = (b+ku)b^n \quad \text{for } k = 1, \ldots, n-1 \tag{2.5}
\]

Setting \( k = 1 \), then \( k = 2, \ldots \), and finally \( k = n-1 \) in (2.5), we obtain

\[
\begin{align*}
b^{n+1} + A_1 b + A_2 b + \ldots + A_{n-1} b + u^n b &= b^{n+1} + ub^n \\
b^{n+1} + 2A_1 b + 2^2 A_2 b + \ldots + 2^{n-1} A_{n-1} b + (2u)^n b &= b^{n+1} + (2u)b^n \\
&\vdots \\
(n-1)A_1 b + (n-2) A_2 b + \ldots + (n-1)^2 A_{n-1} b + (n-1)^n b &= b^{n+1} + ((n-1)u)b^n,
\end{align*}
\]

where each \( A_i \) is a sum of terms each of which is a product in which \( u \) appears exactly \( i \) times and \( b \) appears exactly \( (n-i) \) times. Hence, by (2.4) and (2.6), we get

\[
\begin{align*}
A_1 b + A_2 b + \ldots + A_{n-1} b &= 0 \\
2A_1 b + 2^2 A_2 b + \ldots + 2^{n-1} A_{n-1} b &= 0 \\
&\vdots \\
(n-1)A_1 b + (n-2) A_2 b + \ldots + (n-1)^2 A_{n-1} b &= 0.
\end{align*}
\]

The determinant \( \Delta \) of the matrix of coefficients of the system of linear equations in \( A_1 b, A_2 b, \ldots, A_{n-1} b \) in (2.7) is a Vandermonde determinant, and hence \( \Delta = a \) product of positive integers each of which is less than \( n \). \( \tag{2.8} \)

Moreover, it can be seen that \( \Delta(A_1 b) = 0 \). A similar argument also shows that \( \Delta(b A_1) = 0 \), and hence \( \Delta[A_1, b] = 0 \). Recalling the definition of \( A_i \), we see that

\[
A_i = u b^{n-1} + bu b^{n-2} + \ldots + b^{n-1} u,
\]

and hence

\[
0 = \Delta[A_1, b] = \Delta[u, b^n].
\]
Since \( u = a + b \), [see (2.3)], therefore the above equation yields

\[
\Delta[a,b^n] = 0, \quad (a \in N, \ b \notin N). \tag{2.9}
\]

Combining (2.9) and (2.1), keeping (i) in mind, we see that

\[
0 = \Delta[a,b-b^n] = \Delta[a,b] = \Delta[a,b^n] = \Delta[a,b],
\]

and hence

\[
\Delta[a,b] = 0, \quad (a \in N, \ b \notin N). \tag{2.10}
\]

Now, combining (2.10) and (2.8), we obtain (taking into account repeated factors of \( \Delta \)),

\[
(n!)^\sigma [a,b] = 0 \quad \text{for some positive integer } \sigma,
\]

which proves (2.2) in this case also. Thus completes the proof of (2.2).

Returning to the proof of the theorem, note that if \( \sigma > 1 \) then (2.2) implies that

\[
n![a, (n!)^\sigma b] = 0,
\]

and hence by (iii), \([a, (n!)^\sigma b] = 0\), that is, \((n!)^\sigma(a,b) = 0\). Continuing this process, we eventually obtain \([a,b] = 0\) for all \( a \in N, \ b \notin N. \) But, since \( N \) is commutative, by (i), therefore,

\[
[a,b] = 0 \quad \text{for all } a \in N, \ b \in R. \tag{2.11}
\]

Combining (2.1) and (2.11), we see that

\[
x - x^n \text{ is in the center of } R, \quad \text{for all } x \text{ in } R,
\]

and hence \( R \) is commutative, by a well known theorem of Herstein [1]. This proves the theorem.

**COROLLARY 1.** Let \( R \) be a ring, \( N \) the set of nilpotents, and \( n > 1 \) a fixed integer. Suppose that (i) \( N \) is commutative; (ii) If \( x \notin N \), then \( x^N = x \); (iii) For \( a \in N \) and \( b \in R \), if \( n![a,b] = 0 \), then \([a,b] = 0. \) Then \( R \) is commutative.

As a further corollary, we obtain Jacobson's Theorem [2]:

**COROLLARY 2.** Let \( R \) be a ring and suppose \( n > 1 \) is a fixed integer such that \( x^n = x \) for all \( x \) in \( R. \) Then \( R \) is commutative.

We conclude with the following examples which show that our Main Theorem need not be true if, in hypothesis (iii), "\( n! " \) is replaced by "\( n \)," or if any one of the
hypotheses (i), (ii), (iii) is deleted.

EXAMPLE 1. Let

\[ R = \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix}, \quad a, b, c \in \text{GF}(4) \]

Observe that \( R \) satisfies hypothesis (i) of our Main Theorem, and also satisfies hypothesis (ii) with \( n = 7 \). But hypothesis (iii) is not satisfied for this value of \( n \). However, if \( n! \) is replaced by \( n \) in hypothesis (iii), then \( R \) would satisfy this new hypothesis, \( (n = 7) \). This example shows that "\( n! \)" cannot be replaced by "\( n \)" in (iii).

EXAMPLE 2. Let

\[ R = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad a, b, c \in \text{GF}(3), n=2. \]

It is easily checked that \( R \) satisfies all the hypotheses of our Main Theorem except hypothesis (i), but \( R \) is not commutative. Hence, (i) cannot be deleted.

EXAMPLE 3. Let \( R \) be the ring of quaternions, and let \( n > 1 \) be any positive integer. Note that \( R \) satisfies all the hypotheses of our Main Theorem except hypothesis (ii). Hence, (ii) cannot be deleted.

EXAMPLE 4. Let

\[ R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, 0,1 \in \text{GF}(2) \right\}, n=2. \]

It is readily verified that all the hypotheses of our Main Theorem are satisfied except hypothesis (iii). Hence, (iii) cannot be deleted, since \( R \) is not commutative.

REFERENCES

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems).

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<thead>
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<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
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<td>First Round of Reviews</td>
<td>March 1, 2009</td>
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<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
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</table>

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