By means of Minimax theory, we study the existence of one nontrivial solution and multiple nontrivial solutions for a fourth-order semilinear elliptic problem with Navier boundary conditions.

1. Introduction

Let us consider the problem

\[
\Delta^2 u + c \Delta u = f(x,u), \quad x \in \Omega,
\]
\[
u = 0, \quad \Delta u = 0, \quad x \in \partial \Omega,
\]

where, \(\Delta^2\) is the biharmonic operator and \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with smooth boundary \(\partial \Omega\). This fourth-order semilinear elliptic problem can be considered as an analogue of a class of second order problems which have been studied by many authors. In [3], there was a survey of results obtained in this direction.

Known results about \((P)\) were concerned with the case \(c < \lambda_1\) the first eigenvalue of \(-\Delta\) in \(H^1_0(\Omega)\). In [8], the author proved the existence of a negative solution of \((P)\) by a degree theory with \(f(x,u) = b[(u+1)^{p+1} - 1]\). [4] showed that there existed multiple solutions for \(f(x,u) = bg(x,u)\) by using variational approach. Our recent work obtained a positive solution and a negative solution of \((P)\) by Mountain Pass Theorem, and one more nontrivial solution by Morse theory. It is natural to ask what additional phenomena if \(c\) goes beyond \(\lambda_1\). In [5], the author considered the problem \((P)\) with \(f(x,u) = bg(x,u)\), and got two solutions by using a “variation of linking” theorem under certain conditions.

In the present work, we study the problem \((P)\) with \(c \geq \lambda_1\) by using variational approach.

In Section 2, we prove the existence of one nontrivial solution by Linking Theorem including the Saddle Point Theorem, whether \(c\) is one of the eigenvalues \(\lambda_k\) of \((-\Delta, H^1_0(\Omega))\) or not. In Section 3, we obtain two nontrivial solutions by using a “variation of linking” theorem. Section 4 is devoted to prove the multiplicity of nontrivial solutions, by using the pseudo-index theory. Of course, our results are still valid for second-order semilinear elliptic problem under weaker conditions.
Let $\rho > r$. The Linking Theorem fails, we will apply the Linking Theorem to obtain the weak solution of \( P \) (and even a positive solution). However, if $c \geq \lambda_1$, our previous mechanism fails, we will apply the Linking Theorem to obtain the weak solution of \( P \).

\[ \text{Linking Theorem [9, Theorem 2.12]. Let } X = Y \oplus Z \text{ be a Banach space with } \dim Y < \infty. \text{ Let } \rho > r > 0 \text{ and } z \in Z \text{ such that } \|z\| = r. \text{ Define} \]

\[ M = \{ u = y + \lambda z : \|u\| \leq \rho, \lambda > 0, y \in Y \}, \]

\[ M_0 = \{ u = y + \lambda z : y \in Y, \|u\| = \rho \text{ and } \lambda > 0 \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0 \}, \]

\[ N = \{ u \in Z : \|u\| = r \}. \]

Let $J \in C^1(X, \mathbb{R})$ be such that

\[ b := \inf_{M} J > a := \max_{M} J. \] (2.3)

If $J$ satisfies the (PS) condition, then $J$ has a critical point whose critical value not smaller than $b$.

Assume $\lambda_n < c < \lambda_{n+1}$, $n \geq 1$. Let

\[ Y := \text{span} \{ e_1, \ldots, e_n \}, \quad Z := \{ u \in H : (u, v) = 0, \forall v \in Y \}. \] (2.4)

Since $c < \lambda_{n+1}$, $\|w\|^2 = \int_{\Omega} (|\Delta w|^2 - c|\nabla w|^2)$ and $\|w\|^2_H$ are norms equivalent in the space $Z$, denote $\|w\|^2 = \|w\|^2_H$ for convenience. Then $H = Y \oplus Z$.

The conditions imposed on $f(x, t)$ are as follows:

\[ (f_1) \text{ f : } \Omega \times \mathbb{R} \rightarrow \mathbb{R} \text{ is a Carathéodory function, and for some } 1 < p < 2^* = (2N/N - 4), \]

\[ c_0 > 0, \quad |f(x, u)| \leq c_0 (1 + |u|^{p-1}). \] (2.5)

\[ (f_2) \text{ There exists } \alpha > 2, \text{ for } |u| \gg 1, \]

\[ 0 < \alpha F(x, u) \leq uf(x, u). \] (2.6)

\[ (f_3) f(x, u) = o(|u|), |u| \rightarrow 0 \text{ uniformly on } \Omega. \]
(f4) \((\Lambda_n/2)u^2 \leq F(x,u) = \int_0^u f(x,t)dt\).

**Lemma 2.1.** Under (f1)-(f2), any sequence \((u_n) \subset H\) such that
\[
d := \sup J_c(u_n) < \infty, \quad J'_c(u_n) \rightharpoonup 0, \tag{2.7}
\]
contains a convergent subsequence.

**Proof.** First of all, we observe that
\[
\nabla J_c(u) = u + i^*((1+c)\Delta u - f(x,u)), \tag{2.8}
\]
where, \(i^* : L^2(\Omega) \rightarrow H\) is a compact operator (\(i^*\) is the adjoint of the immersion \(i : H \hookrightarrow L^2(\Omega)\)).

It is enough to prove that \((\|u_n\|)_{n \in \mathbb{N}}\) is bounded, because of (2.8) and (f1). We consider the case \(N \geq 5\). Form (f2), we obtain the existence \(c_1 > 0\) such that
\[
c_1(|u|^{\alpha-1}) \leq F(x,u). \tag{2.9}
\]
Let \(\beta \in (\alpha^{-1},2^{-1})\), for \(n\) large enough and \(c_2, c_3 > 0\), we have
\[
d + 1 + \|u_n\|_H \geq J_c(u_n) - \beta J'_c(u_n, u_n) \geq \int_\Omega \left[ \left( \frac{1}{2} - \beta \right) \left( |\triangle u_n|^2 - c |\nabla u_n|^2 \right) + \beta f(x,u_n) u_n - F(x,u_n) \right] dx \\
\geq \left( \frac{1}{2} - \beta \right) \left( \|z_n\|_H^2 + \Lambda_1 \|y_n\|_2^2 \right) + (\alpha \beta - 1) \int_\Omega F(x,u_n) dx - c_2 \\
\geq \left( \frac{1}{2} - \beta \right) \left( \|z_n\|_H^2 + \Lambda_1 \|y_n\|_2^2 \right) + c_1 (\alpha \beta - 1) |u_n|^{\alpha} - c_3, \tag{2.10}
\]
where, \(u_n = y_n + z_n, y_n \in Y, z_n \in Z\). It is easy to verify that \((u_n)\) is bounded in \(H\) using the fact that \(\text{dim } Y\) is finite.

A standard argument shows that \(\{u_n\}\) has a convergent subsequence in \(H\). Therefore, \(J\) satisfies the \((PS)\) condition. \(\Box\)

**Theorem 2.2.** Assume (f1)–(f4), then problem (P) has at least one nontrivial solution.

**Proof.** (1) We consider the case \(N \geq 5\). We will verify the assumptions of the Linking Theorem. The \((PS)\) condition follows form the preceding Lemma 2.1.

(2) By (f1) (f2), we have
\[
\forall \varepsilon > 0, \exists c_\varepsilon > 0 \text{ such that } |F(x,u)| \leq \varepsilon |u|^2 + c_\varepsilon |u|^p. \tag{2.11}
\]
On \(Z\), we obtain
\[
J_c(u) \geq \frac{1}{2} \|u\|_H^2 - \int_\Omega (\varepsilon |u|^2 + c_\varepsilon |u|^p) = \frac{1}{2} \|u\|_H^2 - \varepsilon |u|^2 - c_\varepsilon |u|^p. \tag{2.12}
\]
By Sobolev imbedding theorem, there exists \( r > 0 \), such that

\[
\inf_{\|u\|_H = r} J_c(u) > 0, \quad u \in Z.
\]  

(2.13)

(3) By \((f_4)\), on \( Y \) we have

\[
J_c(u) \leq \int_{\Omega} \left[ \frac{1}{2} \Lambda_n u^2 - F(x, u) \right] \leq 0.
\]  

(2.14)

Define \( z := re_{n+1}/\|e_{n+1}\|_H \). It follows (2.9), for \( u = y + \lambda z \) with \( \lambda > 0 \), we deduce

\[
J_c(u) = \frac{1}{2} \int_{\Omega} \left[ |\triangle(y + \lambda z)|^2 - c |\nabla(y + \lambda z)|^2 \right] - \int_{\Omega} F(x, u)
\]

\[
\leq \frac{1}{2} \Lambda_n \int_{\Omega} y^2 + \frac{1}{2} \Lambda_n \int_{\Omega} \frac{\lambda}{2} r - c_1 |u|^a_\alpha + c_1 |\Omega|.
\]  

(2.15)

Since on the finite dimensional space \( Y \oplus \mathbb{R}z \) all norms are equivalent, then we get

\[
J_c(u) \rightarrow -\infty, \quad \|u\|_H \rightarrow \infty, \quad u \in Y \oplus \mathbb{R}z.
\]  

(2.16)

Thus, there exists \( \rho > r \) such that

\[
\max_{M_0} J_c = 0,
\]  

(2.17)

where \( M_0 \) is as above. By the Linking Theorem, there exists a critical point \( u \) of \( J \) satisfying \( J_c(u) \geq b > 0 \). Since \( J_c(0) = 0 \), then \( u \) is a nontrivial solution of \((P)\).

Remark 2.3. If \( c < \lambda_1 \), it suffices to use the Mountain Pass Theorem [6, Theorem 2.2] instead of Linking Theorem.

**Theorem 2.4.** Under \((f_1)-(f_3)\) with \( c < \lambda_1 \), \((P)\) has a nontrivial solution.

**Proof.** Please see [6, Theorem 2.15] for its proof in detail, where \( E = H^2 \cap H^1_0(\Omega), (u, v)_E = \int_{\Omega}(\triangle u \triangle v - c \nabla u \nabla v) \) and \( I_c(u) = (1/2) \int_{\Omega}(|\triangle u|^2 - c |\nabla u|^2) - \int_{\Omega} F(x, u) \).

Similarly, we can obtain the following corollary.

**Corollary 2.5.** Under \((f_1)-(f_3)\) and

\((f_4) \quad f(x, t)t \geq 0 \) for all \( t \in \mathbb{R},\)

problem \((P)\) has a positive solution and a negative solution.
Proof. By the truncation technique and the Mountain Pass theorem, the problem
\[ \triangle^2 u + c \triangle u = \bar{f}(x,u), \quad x \in \Omega, \]
\[ u = 0, \quad \triangle u = 0, \quad x \in \partial \Omega, \]
where
\[ \bar{f}(x,u) = \begin{cases} f(x,u), & u \geq 0, \\ 0, & u < 0, \end{cases} \]
has a solution \( u \not\equiv 0 \) satisfying
\[ \int_{\Omega} \Delta u \Delta v - c \nabla u \nabla v = \int_{\Omega} \bar{f}(x,u)v, \quad \forall v \in H. \] (2.19)
Let \( A = \{ x \in \Omega \mid u(x) < 0 \} \), then by the definition of \( \bar{f} \),
\[ \Delta^2 u + c \Delta u = 0, \quad x \in A, \]
\[ u = 0, \quad \Delta u = 0, \quad x \in \partial A. \] (2.20)
By the maximum principle, we have \( u \equiv 0 \) in \( A \), therefore \( A = \emptyset \). Thus \( u \geq 0 \) a.e. on \( \Omega \).
From \((f_4)'\), further using the strong maximum principle [2], we deduce \( u > 0 \), that is, \( u \) is the positive solution of \((P)\). \( \square \)

While for \( \lambda_1 < c \in (\lambda_n, \lambda_{n+1}) \), by the Linking Theorem, we have the following theorem.

**Theorem 2.6.** Under \((f_1)-(f_4)'\), problem \((P)\) has at least a nontrivial solution.

**Proof.** Condition \((f_4)'\) is stronger than \((f_4)\), which is also applied to show that, for \( u \in Y \)
\[ J_c(u) = \frac{1}{2} \int_{\Omega} [ | \Delta u |^2 - c | \nabla u |^2 ] - \int_{\Omega} F(x,u) \leq \frac{1}{2} \Lambda_n \int_{\Omega} u^2 - \int_{\Omega} F(x,u) \leq 0. \] (2.21)
As the similar proof of Theorem 2.2, we obtain the result. \( \square \)

**Remark 2.7.** In Corollary 2.5, we obtain a positive solution of \((P)\) by using truncation technique, if \( c < \lambda_1 \). However, if \( c \geq \lambda_1 \), we cannot expect a positive solution of \((P)\). Indeed, if \( v_1 \) is the eigenfunction corresponding to \( \lambda_1 \), we can assume \( v_1 > 0 \) in \( \Omega \). Therefore, if \( u \) is a solution of \((P)\), we get
\[ \int_{\Omega} f(x,u)v_1 = \int_{\Omega} (\Delta^2 u + c \Delta u)v_1 = \int_{\Omega} (\Delta^2 v_1 + c \Delta v_1)u = \Lambda_1 \int_{\Omega} v_1 u. \] (2.22)
If \( u \) is positive in \( \Omega \), the left-hand side of (2.22) is nonnegative by \((f_4)'\), while the right-hand side is nonpositive, since \( c \geq \lambda_1 \). Thus, there can only be a positive solution \( u(x) \) if \( c = \lambda_1 \), and \( p(x,u(x)) \equiv 0 \).
If \( c = \lambda_k < \lambda_{k+1} \), we can apply the Saddle Point Theorem to obtain a nontrivial solution of \((P)\).
Let $E = V \oplus X$, where $V$ is a real Banach space and $V \ni \neq \{0\}$ is finite dimensional. Suppose $J \in C^1(E, \mathbb{R})$ satisfies (PS) condition, and

$(I_1)$ there is a constant $\alpha$ and a bounded neighborhood $D$ of 0 in $V$, such that $J|_{\partial D} \leq \alpha$,  
$(I_2)$ there is a constant $\beta > \alpha$ such that $J|_X \geq \beta$.

Then $J$ possessed a critical point whose critical value $c \geq \beta$.

**Theorem 2.8.** Under the following conditions

(i) $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, and for some $C > 0$,
$$|f(x,t)| \leq C,$$
(ii) $F(x,\xi) = \int_0^\xi f(x,t)dt \rightarrow \infty$ as $|\xi| \rightarrow \infty$ uniformly for $x \in \Omega$,

problem $(P)$ possesses a nontrivial solution.

**Proof.** Since (i), $J_c$ is of $C^1$. Let $V := \text{span}\{e_1, \ldots, e_k\}$, and $X := \text{span}\{e_j| j \geq k+1\}$, so $X = V^\perp$. Therefore $H = V \oplus X$. We will show that $J_c$ satisfies (i) (ii) and (PS) condition. Then our result follows from the Saddle Point Theorem.

By (i), let $M := \sup_{x \in \Omega, \xi \in \mathbb{R}} |f(x,\xi)|$, then
$$\left| \int_{\Omega} F(x,u)dx \right| \leq M \int_{\Omega} |u|dx \leq M_1 \|u\|_H,$$
(2.23)
for all $u \in H$ via the Hölder and Poincaré inequality. On $X$, the norms $\|u\|^2 = \int_{\Omega} (|\triangle u|^2 - c|\nabla u|^2)$ and $\|u\|^2_2$ are equivalent, we have
$$J_c(u) = \frac{1}{2} \int_{\Omega} \left( |\triangle u|^2 - c|\nabla u|^2 \right) - \int_{\Omega} F(x,u) \geq c_1 \|u\|^2_2 - M_1 \|u\|_H, \quad c_1 > 0,$$
(2.24)
which shows $J_c$ is bounded from below on $X$, that is, (I2) holds.

Next, if $u \in V$, then $u = u^0 + u^-$, where $u^0 \in E^0 := \text{span}\{e_j| \lambda_j = c\}$, and $u^- \in E^- := \text{span}\{e_j| \lambda_j < c\}$. Then
$$J_c(u) = \frac{1}{2} \int_{\Omega} \left( |\triangle u^-|^2 - c|\nabla u^-|^2 \right) - \int_{\Omega} F(x,u^0) - \int_{\Omega} (F(x,u^0 + u^-) - F(x,u^0)).$$
(2.25)
Estimating the last term as in (2.23), since all norms are equivalent on the finite dimensional subspace $E^-$, we have
$$J_c(u) \leq -M_2 \|u^-\|^2_H - \int_{\Omega} F(x,u^0) + M_1 \|u^-\|_H.$$
(2.26)
Now, (2.26) and (ii) show $J_c(u) \to -\infty$ as $u \to \infty$ in $V$. Hence $J_c$ satisfies (I1).
Lastly to verify \((PS)\) condition, it suffices to show that \(|J_c(u_m)| \leq K\) and \(J'_c(u_m) \to 0\) implies \((u_m)\) is bounded, since (i) and (2.8). Writing \(u_m = u_m^0 + u_-^m + u_+^m\), where \(u_m^0 \in E^0\), \(u_-^m \in E^-\), \(u_+^m \in X\). For large \(m\),

\[
|J'_c(u_m)u_+^m| = \int_\Omega \left[ |\Delta u_m \Delta u_+^m| - c \nabla u_m \nabla u_+^m - f(x, u_m) u_+^m \right] dx \leq \|u_+^m\|_H. \tag{2.27}
\]

Consequently, since \(X = V^\perp\), by (2.27) and an estimate like (2.23), we get

\[
\|u_+^m\|_H \geq \|u_+^m\|_H^2 - M_1 \|u_+^m\|_H. \tag{2.28}
\]

which shows that \(\{\|u_+^m\|_H\}\) is bounded. Similarly \(\{\|u_-^m\|_H\}\) is bounded. Finally we claim that \(\{\|u_0^m\|_H\}\) is bounded. Then \((u_m)\) is bounded in \(H\) and we are through. Indeed,

\[
K \geq |J_c(u_m)| = \int_\Omega \left\{ \frac{1}{2} \left[ |\Delta u_+^m|^2 + |\Delta u_-^m|^2 - c \|\nabla u_+^m\|^2 - c \|\nabla u_-^m\|^2 \right] 
- (F(x, u_m) - F(x, u_0^m)) \right\} dx
- \int_\Omega F(x, u_0^m) dx \right\}. \tag{2.29}
\]

By what has already been shown, the first term on the right is bounded independently of \(m\). Therefore

\[
K \geq \left| \int_\Omega F(x, u_0^m) \right| - K_1, \tag{2.30}
\]

so \((\int_\Omega F(x, u_0^m) dx)\) is bounded, which implies \((u_0^m)\) is bounded as the proof of Lemma 4.21 [6]. \(\square\)

**Remark 2.9.** If (ii) is replaced by \(F(x, \xi) \to -\infty\) as \(|\xi| \to \infty\), the above arguments can easily be modified to handle this case.

### 3. The existence of two nontrivial solution

By using the following “a variation of linking” theorem, we can obtain at least two solutions of \((P)\).

**Theorem 3.1** (“a variation of linking”) [7, Corollary 2.4]. Let \(N\) be a subspace of a Hilbert space \(H\), such that \(0 < \dim N < \infty\), and \(M = N^\perp\). Assume \(J\) is a continuously differentiable functional on \(H\), which satisfies for some \(\alpha < \beta\), \(0 < \delta < R\) and \(w_0 \in M \setminus \{0\},\)

\[
J(v) \leq \alpha, \quad v \in N, \quad \|v\| \leq R, \\
J(sw_0 + v) \leq \alpha, \quad s > 0, \quad v \in N, \quad \|sw_0 + v\| = R, \tag{3.1}
\]

\[
J(w) \geq \beta, \quad w \in H, \quad \|w\| = \delta.
\]

If \(J\) satisfies the \((PS)\) condition, then there are at least two solutions of \(J'(u) = 0\), one satisfies \(J(u) \leq \alpha\) and the other \(J(u) \geq \beta\).
Theorem 3.2. Assume \( c \in (\lambda_{l-1}, \lambda_l) \) with \( l \geq 2 \), under the conditions (f1) (f2) and (f3) \( F(x,t) = \int_0^t f(x,s)ds \) satisfies

\[
\frac{1}{2} \Lambda_{l-1} t^2 - w_0(x) \leq F(x,t) \leq \frac{1}{2} \nu_1 t^2 + V(x)^p |t|^p + w_1(x),
\]

where \( \nu_1 < \Lambda_l, p > 2 \).

\( B_j := \int_\Omega w_j(x)dx < \infty, \quad j = 0,1, \)

\[ |Vu|_p \leq C \|u\|_p H, \quad u \in H, \]  

(f6) the following inequality holds:

\[
B_0 + B_1 < \frac{1}{2} \left( 1 - \frac{2 - p}{2} \right) \left( 1 - \frac{\nu_1}{\Lambda_l} \right)^{p/(p-2)} \left( \frac{1}{pC} \right)^{2/(p-2)}, \]

problem (P) has at least two nontrivial solutions.

Proof. Note that above conditions allow \( f(x,0) \neq 0 \).

Under (f1), it is readily checked that the functional \( J_c \) is of \( C^1 \). Let \( N \) be the subspace spanned by the eigenfunctions corresponding to the eigenvalues \( \Lambda_1, \ldots, \Lambda_{l-1} \), and let \( M = N^\perp \cap H \), the orthogonal complement of \( N \) in \( H \). On \( M \) we have by (f5),

\[
J_c(w) = \frac{1}{2} \|w\|_H^2 - \int_\Omega F(x,u) \geq \frac{1}{2} \|w\|_H^2 - \frac{\nu_1}{2} \|w\|_2^2 - |Vu|_p^p - B_1
\]

\[
\geq \frac{1}{2} \left( 1 - \frac{\nu_1}{\Lambda_l} \right) \|w\|_H^2 - C \|w\|_p^p - B_1. \]

If we take \( \delta^{p-2} = \left( 1 - (\nu_1/\Lambda_l) \right)/pC \), we get

\[
J_c(w) \geq \frac{1}{2} \left( 1 - \frac{2 - p}{p} \right) \left( 1 - \frac{\nu_1}{\Lambda_l} \right) \delta^2 - B_1 = \frac{1}{2} \left( 1 - \frac{2 - p}{p} \right) \left( 1 - \frac{\nu_1}{\Lambda_l} \right)^{p/(p-2)} \left( \frac{1}{pC} \right)^{2/(p-2)} - B_1,
\]

\[ \|w\|_H = \delta, \quad w \in M. \]  

On the other hand, on \( N \) we have by (f5),

\[
J_c(v) \leq \frac{1}{2} \int_\Omega (|\Delta v|^2 - c|\nabla v|^2) - \frac{1}{2} \Lambda_{l-1} \int_\Omega v^2 + B_0 \leq B_0, \quad v \in N.
\]

Let \( w_0 \) be an eigenfunction corresponding to the eigenvalue \( \lambda_l \) with unit norm, and let \( N_1 \)
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denote the subspace spanned by $N$ and $w_0$. On $N_1$ we have by $(f_2)$,

$$J_c(u) \leq \frac{1}{2} \int_{\Omega} \left[ |\nabla (v + sw_0)|^2 - c |\nabla (v + sw_0)|^2 \right] - \int_{\Omega} F(x,u)$$

$$\leq \frac{1}{2} \Lambda_{l-1} \int_{\Omega} v^2 + \frac{s}{2} - c_1 |u|^2 + c_1 |\nabla (v + sw_0)|^2 - \int_{\Omega} F(x,u) \leq \frac{1}{2} \Lambda_{l-1} \int_{\Omega} v^2 + \frac{s}{2} - c_1 |u|^2 + c_1 |\nabla (v + sw_0)|^2 - \int_{\Omega} F(x,u) \leq \frac{1}{2} \Lambda_{l-1} \int_{\Omega} v^2 + \frac{s}{2} - c_1 |u|^2 + c_1 |\nabla (v + sw_0)|^2 - \int_{\Omega} F(x,u) \leq \frac{1}{2} \Lambda_{l-1} \int_{\Omega} v^2 + \frac{s}{2} - c_1 |u|^2 + c_1 |\nabla (v + sw_0)|^2$$

where $v \in N$, $s > 0$.

In particular, we see that

$$J_c(u) \longrightarrow -\infty \text{ as } \|u\|_H \longrightarrow \infty, \quad u \in N_1,$$

since all norms are equivalent on the finite dimensional space $N_1$.

Take $R$ so large that $R > \delta$ and

$$J_c(u) \leq B_0, \quad \|u\| \geq R, \quad u \in N_1.$$  \hspace{1cm} (3.10)

If $\beta$ denote the right-hand of (3.5), we see that $B_0 < \beta$ by $(f_6)$. Under $(f_2)$, Lemma 2.1 has shown that $J_c$ satisfies the $(PS)$ condition, then our results will follow from Theorem 3.1, that is, $J_c$ has at least two nontrivial critical points, since $J_c(0) \neq 0$. \hfill $\square$

Remark 3.3. Indeed, if $f(x,0) \neq 0$ in Theorems 2.2, 2.6, and 2.8, we also can obtain at least two nontrivial solutions under certain conditions.

4. The existence of multiple solutions

In this section, we will prove an existence result of multiple solutions by using pseudo-index. We first recall the definition of genus and an abstract theorem of [1].

Let $E$ be a Banach space, $J \in C^1(E, \mathbb{R})$ satisfy $J(-u) = J(u)$ for all $u \in E$. Denote $\Sigma$ to be all the symmetrical and closed sets in $E$, and $\mathbb{Z}_+$ the set of nonnegative integer. Define

$$\gamma(A) := \inf \{ n \in \mathbb{Z}_+: \text{there is a continuous and odd map } \varphi: A \longrightarrow \mathbb{R}^n \setminus \{0\} \}. \hspace{1cm} (4.1)$$

If for all $n \in \mathbb{N}$, there is no such $\varphi$, set $\gamma(A) = +\infty$, while $A = \emptyset$, set $\gamma(A) = 0$. Genus $\gamma$ has the following properties:

Proposition 4.1 [9, Theorem 3.2 IV]. The following conditions hold.

1. If $E = X_1 \bigoplus X_2$, $\dim X_1 = k$, $\gamma(A) > k$, then $A \bigcap X_2 \neq \emptyset$.
2. If $\Omega$ is a symmetrical and bounded neighborhood of 0 in $\mathbb{R}^m$, and there exists a mapping $h \in C(A, \partial \Omega)$ with $h$ an odd homeomorphism, then $\gamma(A) = m$, for $A \in \Sigma$.
3. If $\gamma(A) = k$, $0 \notin A$, then there exist at least $k$ distinct pairs of points in $A$.

Now, we define the pseudo-index $i^*$ by $\gamma$,

$$i^*(A) = \inf_{h \in \Lambda_{*,(0)}} \gamma \left( A \bigcap h(B_1) \right), \hspace{1cm} (4.2)$$

where $A \in \Sigma^* = \{ A \in \Sigma : A \text{ is compact} \}$ and $\Lambda_{*,(0)} = \{ h \in C(E,E) : h \text{ is an odd homeomorphism, for some } \rho > 0, \ h(B_1) \subset J^{-1}(0,\infty) \bigcup B_{\rho} \}$. 

Theorem 4.2 [1, Theorem 3.6 IV]. Let $E$ be a Banach space, $J \in C^1(E, \mathbb{R})$ satisfy $J(-u) = J(u)$ for all $u \in E$. Assume

(I) there exist $\rho, \alpha_0 > 0$ and a subspace $E_1 \subset E$ with $\dim E_1 = m_1$, such that
\[
J|_{E_1} \cap \mathbb{B}_\rho \geq \alpha_0,
\]
(4.3)

(II) there exists a subspace $E_2 \subset E$ with $\dim E_2 = m_2 > m_1$, and $R > 0$ such that $J(u) \leq 0$ for all $u \in E_2 \setminus B_R$,

and $J$ satisfies (PS) condition, then $J$ has at least $m_2 - m_1$ distinct pairs of critical points with critical value
\[
c_n^* = \inf_{i^*(A) \geq n} \sup_{u \in A} J(u).
\]
(4.4)

Theorem 4.3. Assume $(f_1)$, $(f_2)$, $(f_3)$, and

$(f_4)$ $f(x,t)$ is odd in $t$.

If $\lambda_j < c < \lambda_{j+1}$, then $(P)$ has infinitely many nontrivial solutions.

Proof. Let $E = H := H^2 \cap H_0^1(\Omega)$, first we will prove (I) (II) of Theorem 4.2 are satisfied under the conditions of Theorem 4.3.

(I) Let $E_1 := Y_j = \text{span}\{e_1, \ldots, e_j\}$, the similar proof of Theorem 2.2 (2), for all $u \in Y_j^\perp$,
\[
J_c(u) \geq \frac{1}{2} \|u\|^2_H - \varepsilon |u|_2^2 - c_\varepsilon |u|^p_H,
\]
(4.5)

thus, there exist $\rho, \alpha_0 > 0$ small enough, such that
\[
J_c(u) \geq \alpha_0, \quad \forall u \in Y_j^\perp, \quad \|u\|_H = \rho.
\]
(4.6)

(II) For $m \geq 1$ fixed, since all norms are equivalent on the finite dimensional space $Y_{j+m}$, by (2.9) there exists a sufficiently big constant $R > \rho$, such that
\[
J_c(u) \leq \frac{1}{2} \Lambda_{j+m} \int_\Omega u^2 - c_1 |u|^a + c_1 |\Omega|
= c_2 \|u\|^2_H - c_1 \|u\|^a_H + c_1 |\Omega| < 0, \quad u \in Y_{j+m} \setminus B_R.
\]
(4.7)

Next, by the properties of genus and the definition of $c_n^*$, we have
\[
\alpha_0 \leq c_{j+s}^* < +\infty, \quad m \geq s \geq 1.
\]
(4.8)

Indeed, for all $A \in \Sigma^*$ satisfying $i^*(A) \geq j + s$, let $h_0 = \rho \cdot \text{id}$, then $h_0 \in \Lambda_*(\rho)$ and
\[
y(A \cap \partial B_\rho) = y(A \cap h_0(\partial B_1)) \geq \inf_{h \in \Lambda_*(\rho)} y(A \cap h(\partial B_1)) = i^*(A) > j.
\]
(4.9)
By (1°) of Proposition 4.1, $A \cap \partial B_\rho \cap Y_j^* \neq \emptyset$, then (4.6) implies
\[
\sup_{u \in \Lambda} J_c(u) \geq \inf_{u \in \partial B_\rho \cap Y_j^*} J_c(u) \geq \alpha_0.
\] (4.10)

Since $A \in \Sigma^*$ is arbitrary, then $c^*_{j+s} \geq \alpha_0$.

As the proof of Theorem 3.6 IV [1], $c^*_{j+s} < +\infty$, since $j + s \leq \dim Y_{j+m}$.

Thus, we have
\[
\alpha_0 \leq c^*_{j+1} \leq c^*_{j+2} \leq \cdots \leq c^*_{j+m} < +\infty,
\] (4.11)

and the (PS) condition is obtained by Lemma 2.1. Therefore, Theorem 4.2 implies that $J_c$ admits at least $m$ distinct pairs of critical points. Since $m$ is arbitrary and $\lambda_j \to +\infty$ as $j \to \infty$, then (P) has infinitely many nontrivial solutions.

\[\square\]

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**References**


Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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