STRONG CONVERGENCE OF AN ITERATIVE SEQUENCE FOR MAXIMAL MONOTONE OPERATORS IN A BANACH SPACE

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We first introduce a modified proximal point algorithm for maximal monotone operators in a Banach space. Next, we obtain a strong convergence theorem for resolvents of maximal monotone operators in a Banach space which generalizes the previous result by Kamimura and Takahashi in a Hilbert space. Using this result, we deal with the convex minimization problem and the variational inequality problem in a Banach space.

1. Introduction

Let $E$ be a real Banach space and let $T \subset E \times E^*$ be a maximal monotone operator. Then we study the problem of finding a point $v \in E$ satisfying

$$0 \in Tv.$$  \hfill (1.1)

Such a problem is connected with the convex minimization problem. In fact, if $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, then Rockafellar’s theorem [14, 15] ensures that the subdifferential mapping $\partial f \subset E \times E^*$ of $f$ is a maximal monotone operator. In this case, the equation $0 \in \partial f(v)$ is equivalent to $f(v) = \min_{x \in E} f(x)$.

In 1976, Rockafellar [17] proved the following weak convergence theorem.

**Theorem 1.1 (Rockafellar [17]).** Let $H$ be a Hilbert space and let $T \subset H \times H$ be a maximal monotone operator. Let $I$ be the identity mapping and let $J_r = (I + rT)^{-1}$ for all $r > 0$. Define a sequence $\{x_n\}$ as follows: $x_1 = x \in H$ and

$$x_{n+1} = J_{r_n}x_n \quad (n = 1, 2, \ldots),$$  \hfill (1.2)

where $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n \to \infty} r_n > 0$. If $T^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element of $T^{-1}0$.

This is called the proximal point algorithm, which was first introduced by Martinet [11]. If $T = \partial f$, where $f : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function,
then (1.2) is reduced to

\[ x_{n+1} = \arg\min_{y \in H} \left\{ f(y) + \frac{1}{2r_n} \|x_n - y\|_2^2 \right\} \quad (n = 1, 2, \ldots). \quad (1.3) \]

Later, many researchers studied the convergence of the proximal point algorithm in a Hilbert space; see Brézis and Lions [4], Lions [10], Passty [12], Güler [7], Solodov and Svaiter [19] and the references mentioned there. In particular, Kamimura and Takahashi [8] proved the following strong convergence theorem.

**Theorem 1.2 (Kamimura and Takahashi [8])**. Let \( H \) be a Hilbert space and let \( T \subset H \times H \) be a maximal monotone operator. Let \( J_r = (I + rT)^{-1} \) for all \( r > 0 \) and let \( \{x_n\} \) be a sequence defined as follows: \( x_1 = x \in H \) and

\[ x_{n+1} = \alpha_n x + (1 - \alpha_n)J_{r_n}x_n \quad (n = 1, 2, \ldots), \quad (1.4) \]

where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \) and \( \lim_{n \to \infty} r_n = \infty. \) If \( T^{-1}0 \neq \emptyset, \) then the sequence \( \{x_n\} \) converges strongly to \( P_{T^{-1}0}(x), \) where \( P_{T^{-1}0} \) is the metric projection from \( H \) onto \( T^{-1}0. \)

Recently, using the hybrid method in mathematical programming, Kamimura and Takahashi [9] obtained a strong convergence theorem for maximal monotone operators in a Banach space, which extended the result of Solodov and Svaiter [19] in a Hilbert space. On the other hand, Censor and Reich [6] introduced a convex combination which is based on Bregman distance and studied some iterative schemes for finding a common asymptotic fixed point of a family of operators in finite-dimensional spaces.

In this paper, motivated by Censor and Reich [6], we introduce the following iterative sequence for a maximal monotone operator \( T \subset E \times E^* \) in a smooth and uniformly convex Banach space: \( x_1 = x \in E \) and

\[ x_{n+1} = J^{-1}(\alpha_n Jx + (1 - \alpha_n)JJ_{r_n}x_n) \quad (n = 1, 2, \ldots), \quad (1.5) \]

where \( \{\alpha_n\} \subset [0, 1], \{r_n\} \subset (0, \infty), J \) is the duality mapping from \( E \) into \( E^*, \) and \( J_r = (J + rT)^{-1}J \) for all \( r > 0. \) Then we extend Kamimura-Takahashi’s theorem to the Banach space (Theorem 3.3). It should be noted that we do not assume the weak sequential continuity of the duality mapping [1, 5, 13]. Finally, we apply Theorem 3.3 to the convex minimization problem and the variational inequality problem.

2. Preliminaries

Let \( E \) be a (real) Banach space with norm \( \| \cdot \| \) and let \( E^* \) denote the Banach space of all continuous linear functionals on \( E. \) For all \( x \in E \) and \( x^* \in E^*, \) we denote \( x^*(x) \) by \( \langle x, x^* \rangle. \) We denote by \( \mathbb{R} \) and \( \mathbb{N} \) the set of all real numbers and the set of all positive integers, respectively. The **duality mapping** \( J \) from \( E \) into \( E^* \) is defined by

\[ J(x) = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = ||x^*||^2 \} \quad (2.1) \]
for all $x \in E$. If $E$ is a Hilbert space, then $J = I$, where $I$ is the identity mapping. We sometimes identify a set-valued mapping $A : E \rightarrow 2^{E^*}$ with its graph $G(A) = \{(x,x^*) : x^* \in Ax\}$. An operator $T \subset E \times E^*$ with domain $D(T) = \{x \in E : Tx \neq \emptyset\}$ and range $R(T) = \bigcup \{Tx : x \in D(T)\}$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(x,x^*), (y,y^*) \in T$. We denote the set $\{x \in E : 0 \in Tx\}$ by $T^{-1}0$. A monotone operator $T \subset E \times E^*$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. If $T \subset E \times E^*$ is maximal monotone, then the solution set $T^{-1}0$ is closed and convex. A proper function $f : E \rightarrow (-\infty, \infty]$ (which means that $f$ is not identically $\infty$) is said to be convex if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha) f(y)$$

for all $x, y \in E$ and $\alpha \in (0,1)$. The function $f$ is also said to be lower semicontinuous if the set $\{x \in E : f(x) \leq r\}$ is closed in $E$ for all $r \in \mathbb{R}$. For a proper lower semicontinuous convex function $f : E \rightarrow (-\infty, \infty]$, the subdifferential $\partial f$ of $f$ is defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y) \ \forall \ y \in E\}$$

for all $x \in E$. It is easy to verify that $0 \in \partial f(v)$ if and only if $f(v) = \min_{x \in E} f(x)$. It is known that the subdifferential of the function $f$ defined by $f(x) = \|x\|^2/2$ for all $x \in E$ is the duality mapping $J$. The following theorem is also well known (see Takahashi [21] for details).

**Theorem 2.1.** Let $E$ be a Banach space, let $f : E \rightarrow (-\infty, \infty]$ be a proper lower semicontinuous convex function, and let $g : E \rightarrow \mathbb{R}$ be a continuous convex function. Then

$$\partial(f + g)(x) = \partial f(x) + \partial g(x)$$

for all $x \in E$.

A Banach space $E$ is said to be strictly convex if

$$\|x\| = \|y\| = 1, \ x \neq y \Rightarrow \left\|\frac{x + y}{2}\right\| < 1.$$  

(2.5)  

Also, $E$ is said to be uniformly convex if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \ \|x - y\| \geq \varepsilon \Rightarrow \left\|\frac{x + y}{2}\right\| \leq 1 - \delta.$$  

(2.6)  

It is also said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in \{z \in E : \|z\| = 1\}$. We know the following (see Takahashi [20] for details):

1. if $E$ is smooth, then $J$ is single-valued;
2. if $E$ is strictly convex, then $J$ is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x,x^*),(y,y^*) \in J$ with $x \neq y$;
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(3) if \( E \) is reflexive, then \( J \) is surjective;
(4) if \( E \) is uniformly convex, then it is reflexive;
(5) if \( E^* \) is uniformly convex, then \( J \) is uniformly norm-to-norm continuous on each bounded subset of \( E \).

Let \( E \) be a smooth Banach space. We use the following function studied in Alber [1], Kamimura and Takahashi [9], and Reich [13]:

\[
\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2
\]

(2.8)

for all \( x, y \in E \). It is obvious from the definition of \( \phi \) that \((\|x\| - \|y\|)^2 \leq \phi(x, y) \) for all \( x, y \in E \). We also know that

\[
\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, z - y \rangle
\]

(2.9)

for all \( x, y, z \in E \). The following lemma was also proved in [9].

Lemma 2.2 (Kamimura-Takahashi [9]). Let \( E \) be a smooth and uniformly convex Banach space and let \( \{x_n\} \) and \( \{y_n\} \) be sequences in \( E \) such that either \( \{x_n\} \) or \( \{y_n\} \) is bounded. If

\[
\lim_{n \to \infty} \phi(x_n, y_n) = 0,
\]

then \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).

Let \( E \) be a strictly convex, smooth, and reflexive Banach space, and let \( T \subset E \times E^* \) be a monotone operator. Then \( T \) is maximal if and only if \( R(J + rT) = E^* \) for all \( r > 0 \) (see Barbu [2] and Takahashi [21]). If \( T \subset E \times E^* \) is a maximal monotone operator, then for each \( r > 0 \) and \( x \in E \), there corresponds a unique element \( x_r \in D(T) \) satisfying

\[
J(x) \in J(x_r) + rTx_r.
\]

(2.10)

We define the resolvent of \( T \) by \( J_r x = x_r \). In other words, \( J_r = (J + rT)^{-1} J \) for all \( r > 0 \). The resolvent \( J_r \) is a single-valued mapping from \( E \) into \( D(T) \). If \( E \) is a Hilbert space, then \( J_r \) is nonexpansive, that is, \( \|J_r x - J_r y\| \leq \|x - y\| \) for all \( x, y \in E \) (see Takahashi [20]). It is easy to show that \( T^{-1} 0 = F(J_r) \) for all \( r > 0 \), where \( F(J_r) \) denotes the set of all fixed points of \( J_r \). We can also define, for each \( r > 0 \), the Yosida approximation of \( T \) by \( A_r = (J - J_r)/r \). We know that \( (J_r x, A_r x) \in T \) for all \( r > 0 \) and \( x \in E \). Let \( C \) be a nonempty closed convex subset of the space \( E \). By Alber [1] or Kamimura and Takahashi [9], for each \( x \in E \), there corresponds a unique element \( x_0 \in C \) (denoted by \( P_C(x) \)) such that

\[
\phi(x_0, x) = \min_{y \in C} \phi(y, x).
\]

(2.11)

The mapping \( P_C \) is called the generalized projection from \( E \) onto \( C \). If \( E \) is a Hilbert space, then \( P_C \) is coincident with the metric projection from \( E \) onto \( C \). We also know the following lemma.

Lemma 2.3 ([1], see also [9]). Let \( E \) be a smooth Banach space, let \( C \) be a nonempty closed convex subset of \( E \), and let \( x \in E \) and \( x_0 \in C \). Then the following are equivalent:

1. \( \phi(x_0, x) = \min_{y \in C} \phi(y, x) \);
2. \( \langle y - x_0, Jy - Jx_0 \rangle \leq 0 \) for all \( y \in C \).
3. **Strong convergence theorem**

The resolvents of maximal monotone operators have the following property, which was proved in the case of the resolvents of normality operators in Kamimura and Takahashi [9].

**Lemma 3.1.** Let $E$ be a strictly convex, smooth, and reflexive Banach space, let $T \subset E \times E^*$ be a maximal monotone operator with $T^{-1}0 \neq \emptyset$, and let $J_r = (J + rT)^{-1}J$ for each $r > 0$. Then

$$\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x)$$

(3.1)

for all $r > 0$, $u \in T^{-1}0$, and $x \in E$.

**Proof.** Let $r > 0$, $u \in T^{-1}0$, and $x \in E$ be given. By the monotonicity of $T$, we have

$$\phi(u, x) = \phi(u, J_r x) + \phi(J_r x, x) + 2\langle u - J_r x, JJ_r x - Jx \rangle$$

(3.2)

$$\geq \phi(u, J_r x) + \phi(J_r x, x).$$

□

Let $E$ be a strictly convex, smooth, and reflexive Banach space, and let $J$ be the duality mapping from $E$ into $E^*$. Then $J^{-1}$ is also single-valued, one-to-one, and surjective, and it is the duality mapping from $E^*$ into $E$. We make use of the following mapping $V$ studied in Alber [1]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2$$

(3.3)

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping $g$ defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous and convex function from $E^*$ into $\mathbb{R}$. We can prove the following lemma.

**Lemma 3.2.** Let $E$ be a strictly convex, smooth, and reflexive Banach space, and let $V$ be as in (3.3). Then

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*)$$

(3.4)

for all $x \in E$ and $x^*, y^* \in E^*$.

**Proof.** Let $x \in E$ be given. Define $g(x^*) = V(x, x^*)$ and $f(x^*) = \|x^*\|^2$ for all $x^* \in E^*$. Since $J^{-1}$ is the duality mapping from $E^*$ into $E$, we have

$$\partial g(x^*) = \partial (-2\langle x, \cdot \rangle + f)(x^*) = -2x + 2J^{-1}(x^*)$$

(3.5)

for all $x^* \in E^*$. Hence, we have

$$g(x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq g(x^* + y^*),$$

(3.6)
that is,
\[ V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*) \]  
(3.7)
for all \( x^*, y^* \in E^* \).  
\( \square \)

Now we can prove the following strong convergence theorem, which is a generalization of Kamimura-Takahashi’s theorem (Theorem 1.2).

**Theorem 3.3.** Let \( E \) be a smooth and uniformly convex Banach space and let \( T \subset E \times E^* \) be a maximal monotone operator. Let \( J_r = (J + rT)^{-1}J \) for all \( r > 0 \) and let \( \{x_n\} \) be a sequence defined as follows: \( x_1 = x \in E \) and
\[ x_{n+1} = J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jr_n x_n) \quad (n = 1, 2, \ldots), \]  
(3.8)
where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^\infty \alpha_n = \infty \), and \( \lim_{n \to \infty} r_n = \infty \). If \( T^{-1}0 \neq \emptyset \), then the sequence \( \{x_n\} \) converges strongly to \( Pt_{-10}(x) \), where \( Pt_{-10} \) is the generalized projection from \( E \) onto \( T^{-1}0 \).

**Proof.** Put \( y_n = Jr_n x_n \) for all \( n \in \mathbb{N} \). We denote the mapping \( Pt_{-10} \) by \( P \). We first prove that \( \{x_n\} \) is bounded. From Lemma 3.1, we have
\[
\phi(Px, x_{n+1}) = \phi(Px, J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jr_n x_n)) \\
= V(Px, \alpha_n Jx + (1 - \alpha_n) Jr_n x_n) \\
\leq \alpha_n V(Px, Jx) + (1 - \alpha_n) V(Px, Jr_n x_n) \\
= \alpha_n \phi(Px, x) + (1 - \alpha_n) \phi(Px, Jr_n x_n) \\
\leq \alpha_n \phi(Px, x) + (1 - \alpha_n) \phi(Px, x_n) 
\]  
(3.9)
for all \( n \in \mathbb{N} \). Hence, by induction, we have \( \phi(Px, x_n) \leq \phi(Px, x) \) for all \( n \in \mathbb{N} \). Since \( (\|u\| - \|v\|)^2 \leq \phi(u, v) \) for all \( u, v \in E \), the sequence \( \{x_n\} \) is bounded. Since \( \phi(Px, y_n) = \phi(Px, Jr_n x_n) \leq \phi(Px, x_n) \) for all \( n \in \mathbb{N} \), \( \{y_n\} \) is also bounded. We next prove
\[
\limsup_{n \to \infty} \langle x_n - Px, Jx - JPx \rangle \leq 0. \]  
(3.10)
Put \( z_n = x_{n+1} \) for all \( n \in \mathbb{N} \). Since \( \{z_n\} \) is bounded, we have a subsequence \( \{z_{n_k}\} \) of \( \{z_n\} \) such that
\[
\lim_{k \to \infty} \langle z_{n_k} - Px, Jx - JPx \rangle = \limsup_{n \to \infty} \langle z_n - Px, Jx - JPx \rangle \]  
(3.11)
and \( \{z_{n_k}\} \) converges weakly to some \( v \in E \). From the definition of \( \{x_n\} \), we have
\[
Jz_n - Jy_n = \alpha_n (Jx - Jy_n) \]  
(3.12)
for all \( n \in \mathbb{N} \). Since \( \{y_n\} \) is bounded and \( \alpha_n \to 0 \) as \( n \to \infty \), we have
\[
\lim_{n \to \infty} \|Jz_n - Jy_n\| = \lim_{n \to \infty} \alpha_n \|Jx - Jy_n\| = 0. \]  
(3.13)
Since $E$ is uniformly convex, $E^*$ is uniformly smooth, and hence the duality mapping $J^{-1}$ from $E^*$ into $E$ is uniformly norm-to-norm continuous on each bounded subset of $E^*$. Therefore, we obtain from (3.13) that
\[
\lim_{n \to \infty} \|z_n - y_n\| = \lim_{n \to \infty} \|J^{-1}(Jz_n) - J^{-1}(Jy_n)\| = 0. \tag{3.14}
\]
This implies that $y_{n_i} \rightharpoonup v$ as $i \to \infty$, where $\rightharpoonup$ implies the weak convergence. On the other hand, from $r_n \to \infty$ as $n \to \infty$, we have
\[
\lim_{n \to \infty} \|A_{r_n}x_n\| = \lim_{n \to \infty} \frac{1}{r_n} \|Jx_n - Jy_n\| = 0. \tag{3.15}
\]
If $(z, z^*) \in T$, then it holds from the monotonicity of $T$ that
\[
\langle z - y_{n_i}, z^* - A_{r_{n_i}}x_{n_i} \rangle \geq 0 \tag{3.16}
\]
for all $i \in \mathbb{N}$. Letting $i \to \infty$, we get $\langle z - v, z^* \rangle \geq 0$. Then, the maximality of $T$ implies $v \in T^{-1} v \in T^{-1} 0$. Applying Lemma 2.3, we obtain
\[
\limsup_{n \to \infty} \langle z_n - Px, Jx - JPx \rangle = \lim_{i \to \infty} \langle z_{n_i} - Px, Jx - JPx \rangle = \langle v - Px, Jx - JPx \rangle \leq 0. \tag{3.17}
\]
Finally, we prove that $x_n \rightharpoonup Px$ as $n \to \infty$. Let $\varepsilon > 0$ be given. From (3.10), we have $m \in \mathbb{N}$ such that
\[
\langle x_n - Px, Jx - JPx \rangle \leq \varepsilon \tag{3.18}
\]
for all $n \geq m$. If $n \geq m$, then it holds from (3.18) and Lemmas 3.1 and 3.2 that
\[
\phi(Px, x_{n+1}) = V(Px, \alpha_n Jx + (1 - \alpha_n) Jy_n) \\
\leq V(Px, \alpha_n Jx + (1 - \alpha_n) Jy_n - \alpha_n(Jx - JPx)) \\
- 2\langle J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jy_n) - Px, -\alpha_n(Jx - JPx) \rangle \\
= V(Px, (1 - \alpha_n) Jy_n + \alpha_n JPx) + 2\langle x_{n+1} - Px, \alpha_n (Jx - JPx) \rangle \\
\leq (1 - \alpha_n) V(Px, Jy_n) + \alpha_n V(Px, JPx) + 2\alpha_n \langle x_{n+1} - Px, Jx - JPx \rangle \tag{3.19} \\
\leq (1 - \alpha_n) \phi(Px, y_n) + \alpha_n \phi(Px, Px) + 2\alpha_n \varepsilon \\
= (1 - \alpha_n) \phi(Px, J_{r_n} x_n) + 2\alpha_n \varepsilon \\
\leq (1 - \alpha_n) \phi(Px, x_n) + 2\alpha_n \varepsilon \\
= 2\varepsilon \{1 - (1 - \alpha_n)\} + (1 - \alpha_n) \phi(Px, x_n).
Therefore, we have
\[
\phi(Px, x_{n+1}) 
\leq 2\varepsilon \{1 - (1 - \alpha_n)\} + (1 - \alpha_n) \{1 - (1 - \alpha_{n-1})\} + (1 - \alpha_{n-1}) \phi(Px, x_{n-1}) 
= 2\varepsilon \{1 - (1 - \alpha_n)(1 - \alpha_{n-1})\} + (1 - \alpha_n)(1 - \alpha_{n-1}) \phi(Px, x_{n-1}) 
\leq \cdots \leq 2\varepsilon \left\{1 - \prod_{i=m}^{n} (1 - \alpha_i)\right\} + \prod_{i=m}^{n} (1 - \alpha_i) \phi(Px, x_m) 
\tag{3.20}
\]
for all \( n \geq m \). Since \( \sum_{i=1}^{\infty} \alpha_i = \infty \), we have \( \prod_{i=m}^{\infty} (1 - \alpha_i) = 0 \) (see Takahashi [21]). Hence, we have
\[
\limsup_{n \to \infty} \phi(Px, x_n) = \limsup_{l \to \infty} \phi(Px, x_{m+l+1}) 
\leq \limsup_{l \to \infty} \left[ 2\varepsilon \left\{1 - \prod_{i=m}^{m+l} (1 - \alpha_i)\right\} + \prod_{i=m}^{m+l} (1 - \alpha_i) \phi(Px, x_m) \right] = 2\varepsilon. 
\tag{3.21}
\]
This implies \( \limsup_{n \to \infty} \phi(Px, x_n) \leq 0 \) and hence we get
\[
\lim_{n \to \infty} \phi(Px, x_n) = 0. 
\tag{3.22}
\]
Applying Lemma 2.2, we obtain
\[
\lim_{n \to \infty} \|Px - x_n\| = 0. 
\tag{3.23}
\]
Therefore, \( \{x_n\} \) converges strongly to \( P_{T^{-1}0}(x) \).

4. Applications

In this section, we first study the problem of finding a minimizer of a proper lower semicontinuous convex function in a Banach space.

**Theorem 4.1.** Let \( E \) be a smooth and uniformly convex Banach space and let \( f : E \to (-\infty, \infty] \) be a proper lower semicontinuous convex function such that \( (\partial f)^{-1}(0) \neq \emptyset \). Let \( \{x_n\} \) be a sequence defined as follows: \( x_1 = x \in E \) and
\[
y_n = \arg\min_{y \in E} \left\{ f(y) + \frac{1}{2r_n} \|y\|^2 - \frac{1}{r_n} \langle y, Jx_n \rangle \right\} \quad (n = 1, 2, \ldots), 
\]
\[
x_{n+1} = J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jy_n) \quad (n = 1, 2, \ldots), 
\tag{4.1}
\]
where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), and \( \lim_{n \to \infty} r_n = \infty \). Then the sequence \( \{x_n\} \) converges strongly to \( P_{(\partial f)^{-1}(0)}(x) \).

**Proof.** By Rockafellar’s theorem [14, 15], the subdifferential mapping \( \partial f \subset E \times E^* \) is maximal monotone (see also Borwein [3], Simons [18], or Takahashi [21]). Fix \( r > 0 \), \( z \in E \), and let \( J_r \) be the resolvent of \( \partial f \). Then we have
\[
J_z \in J(J_r z) + r\partial f(J_r z). 
\tag{4.2}
\]
and hence,
\[
0 \in \partial f(J_r z) + \frac{1}{r} J(J_r z) - \frac{1}{r} Jz = \partial \left( f + \frac{1}{2r} \| \cdot \|^2 - \frac{1}{r} Jz \right)(J_r z).
\] (4.3)

Thus, we have
\[
J_r z = \arg\min_{y \in E} \left\{ f(y) + \frac{1}{2r} \| y \|^2 - \frac{1}{r} \langle y, Jz \rangle \right\}.
\] (4.4)

Therefore, \( y_n = J_r x_n \) for all \( n \in \mathbb{N} \). Using Theorem 3.3, \( \{x_n\} \) converges strongly to \( P_{(\partial f)^{-1}(0)}(x) \). □

We next study the problem of finding a solution of a variational inequality. Let \( C \) be a nonempty closed convex subset of a Banach space \( E \) and let \( A : C \to E^* \) be a single-valued, monotone, and hemicontinuous operator which is \textit{hemicontinuous}, that is, continuous along each line segment in \( C \) with respect to the weak* topology of \( E^* \). Then a point \( v \in C \) is said to be a solution of the \textit{variational inequality} for \( A \) if
\[
\langle y - v, Av \rangle \geq 0 \quad (4.5)
\]
holds for all \( y \in C \). We denote by \( VI(C, A) \) the set of all solutions of the variational inequality for \( A \). We also denote by \( NC(x) \) the \textit{normal cone} for \( C \) at a point \( x \in C \), that is,
\[
NC(x) = \{ x^* \in E^* : \langle y - x, x^* \rangle \leq 0 \ \forall \ y \in C \}.
\] (4.6)

**Theorem 4.2.** Let \( C \) be a nonempty closed convex subset of a smooth and uniformly convex Banach space \( E \) and let \( A : C \to E^* \) be a single-valued, monotone, and hemicontinuous operator such that \( VI(C, A) \neq \emptyset \). Let \( \{x_n\} \) be a sequence defined as follows: \( x_1 = x \in E \) and
\[
y_n = VI\left( C, A + \frac{1}{r_n} (J - Jx_n) \right) \quad (n = 1, 2, \ldots),
\]
\[
x_{n+1} = J^{-1} \left( \alpha_n Jx + (1 - \alpha_n) Jy_n \right) \quad (n = 1, 2, \ldots),
\] (4.7)

where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), and \( \lim_{n \to \infty} r_n = \infty \). Then, the sequence \( \{x_n\} \) converges strongly to \( P_{VI(C, A)}(x) \).

**Proof.** By Rockafellar’s theorem [16], the mapping \( T \subset E \times E^* \) defined by
\[
Tx = \begin{cases} A(x) + NC(x), & \text{if } x \in C, \\ \emptyset, & \text{otherwise}, \end{cases}
\] (4.8)
is maximal monotone and \( T^{-1}0 = VI(C, A) \). Fix \( r > 0 \), \( z \in E \), and let \( J_r \) be the resolvent of \( T \). Then we have
\[
Jz \in J(J_r z) + r J(J_r z)
\] (4.9)
and hence,
\[-A(Jr,z) + \frac{1}{r}(Jz - J(Jr,z)) \in NC(Jr,z). \tag{4.10}\]
Thus, we have
\[
\left\langle y - Jr,z, A(Jr,z) + \frac{1}{r}(J(Jr,z) - Jz) \right\rangle \geq 0 \tag{4.11}
\]
for all \( y \in C \), that is,
\[
Jr,z = VI \left( C, A + \frac{1}{r}(J - Jz) \right). \tag{4.12}
\]
Therefore, \( y_n = Jr_nx_n \) for all \( n \in \mathbb{N} \). Using Theorem 3.3, \( \{x_n\} \) converges strongly to \( P_{VI(C,A)}(x) \). \qed

References


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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

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