On $m$-sectorial Schrödinger-type operators with singular potentials on manifolds of bounded geometry

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Abstract. We consider a Schrödinger-type differential expression $H_V = \nabla^* \nabla + V$, where $\nabla$ is a $C^\infty$-bounded Hermitian connection on a Hermitian vector bundle $E$ of bounded geometry over a manifold of bounded geometry $(M, g)$ with metric $g$ and positive $C^\infty$-bounded measure $d\mu$, and $V$ is a locally integrable section of the bundle of endomorphisms of $E$. We give a sufficient condition for $m$-sectoriality of a realization of $H_V$ in $L^2(E)$. In the proof we use generalized Kato’s inequality as well as a result on the positivity of $u \in L^2(M)$ satisfying the equation $(\Delta_M + b)u = \nu$, where $\Delta_M$ is the scalar Laplacian on $M$, $b > 0$ is a constant and $\nu \geq 0$ is a positive distribution on $M$.

Keywords: Schrödinger operator, $m$-sectorial, manifold, bounded geometry, singular potential

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1. Introduction and the main result

1.1 The setting. Let $(M, g)$ be a $C^\infty$ Riemannian manifold without boundary, with metric $g$ and $\dim M = n$. We will assume that $M$ is connected. We will also assume that $M$ has bounded geometry. Moreover, we will assume that we are given a positive $C^\infty$-bounded measure $d\mu$, i.e. in any local coordinates $x^1, x^2, \ldots, x^n$ there exists a strictly positive $C^\infty$-bounded density $\rho(x)$ such that $d\mu = \rho(x) dx^1 dx^2 \cdots dx^n$.

Let $E$ be a Hermitian vector bundle over $M$. We will assume that $E$ is a bundle of bounded geometry (i.e. it is supplied by an additional structure: trivializations of $E$ on every canonical coordinate neighborhood $U$ such that the corresponding matrix transition functions $h_{U,U'}$ on all intersections $U \cap U'$ of such neighborhoods are $C^\infty$-bounded, i.e. all derivatives $\partial^\alpha_y h_{U,U'}(y)$, where $\alpha$ is a multiindex, with respect to canonical coordinates are bounded with bounds $C_\alpha$ which do not depend on the chosen pair $U, U'$).

We denote by $L^2(E)$ the Hilbert space of square integrable sections of $E$ with respect to the scalar product

\begin{equation}
(u, v) = \int_M \langle u(x), v(x) \rangle d\mu(x).
\end{equation}

Here $\langle \cdot, \cdot \rangle$ denotes the fiberwise inner product in $E_x$. 
In what follows, \( C^\infty(E) \) denotes smooth sections of \( E \), and \( C_c^\infty(E) \) denotes smooth compactly supported sections of \( E \).

Let

\[
\nabla: C^\infty(E) \to C^\infty(T^*M \otimes E)
\]

be a Hermitian connection on \( E \) which is \( C^\infty \)-bounded as a linear differential operator, i.e. in any canonical coordinate system \( U \) (with the chosen trivializations of \( E|_U \) and \( (T^*M \otimes E)|_U \)), \( \nabla \) is written in the form

\[
\nabla = \sum_{|\alpha| \leq 1} a_\alpha(y) \partial^\alpha_y,
\]

where \( \alpha \) is a multiindex, and the coefficients \( a_\alpha(y) \) are matrix functions whose derivatives \( \partial^\beta_y a_\alpha(y) \) for any multiindex \( \beta \) are bounded by a constant \( C_\beta \) which does not depend on the chosen canonical neighborhood.

We will consider a Schrödinger type differential expression of the form

\[
H_V = \nabla^* \nabla + V,
\]

where \( V \) is a measurable section of the bundle \( \text{End} E \) of endomorphisms of \( E \). Here

\[
\nabla^*: C^\infty(T^*M \otimes E) \to C^\infty(E)
\]

is a differential operator which is formally adjoint to \( \nabla \) with respect to the scalar product \((1.1)\).

If we take \( \nabla = d \), where \( d: C^\infty(M) \to \Omega^1(M) \) is the standard differential, then \( d^*d: C^\infty(M) \to C^\infty(M) \) is called the scalar Laplacian and will be denoted by \( \Delta_M \).

In what follows, we use the notations

\[
(1.2) \quad (\text{Re} V)(x) := \frac{V(x) + (V(x))^*}{2}, \quad (\text{Im} V)(x) := \frac{V(x) - (V(x))^*}{2i}, \quad x \in M,
\]

where \( i = \sqrt{-1} \) and \((V(x))^*\) denotes the adjoint of the linear operator \( V(x): E_x \to E_x \) (in the sense of linear algebra).

By \((1.2)\), for all \( x \in M \), \((\text{Re} V)(x)\) and \((\text{Im} V)(x)\) are self-adjoint linear operators \( E_x \to E_x \), and we have the following decomposition:

\[
V(x) = (\text{Re} V)(x) + i(\text{Im} V)(x).
\]

For every \( x \in M \), we have the following decomposition:

\[
(1.3) \quad (\text{Re} V)(x) = (\text{Re} V)^+(x) - (\text{Re} V)^-(x).
\]

Here \((\text{Re} V)^+(x) = P_+(x)(\text{Re} V)(x)\), where \( P_+(x) := \chi_{[0, +\infty)}((\text{Re} V)(x)) \), and \((\text{Re} V)^-(x) = -P_-(x)(\text{Re} V)(x)\), where \( P_-(x) := \chi_{(-\infty, 0)}((\text{Re} V)(x)) \). Here \( \chi_A \) denotes the characteristic function of the set \( A \).

We make the following assumption on \( V \).
Assumption A.

(i) \((\text{Re } V)^+ \in L^1_{\text{loc}}(\text{End } E), (\text{Re } V)^- \in L^1_{\text{loc}}(\text{End } E)\) and \((\text{Im } V) \in L^1_{\text{loc}}(\text{End } E)\).

(ii) There exists a constant \(L > 0\) such that for all \(u \in L^2(E)\) and all \(x \in M\),

\[
|(|\text{Im } V|(x)||u(x)|^2 \leq L\langle (\text{Re } V)^+ (x)u(x), u(x) \rangle,
\]

where \(|(|\text{Im } V|(x)|\) denotes the norm of the operator \((\text{Im } V)(x): E_x \to E_x\), \(|u(x)|\) denotes the norm in the fiber \(E_x\) and \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(E_x\).

1.2 Sobolev space \(W^{1,2}(E)\). By \(W^{1,2}(E)\) we will denote the completion of the space \(C_\infty^0 (E)\) with respect to the norm \(\| \cdot \|_1\) defined by the scalar product

\[
(u, v)_1 := (u, v) + (\nabla u, \nabla v) \quad u, v \in C_\infty^0 (E).
\]

By \(W^{-1,2}(E)\) we will denote the dual of \(W^{1,2}(E)\).

1.3 Quadratic forms. In what follows, all quadratic forms are considered in the Hilbert space \(L^2(E)\).

1. By \(h_0\) we denote the quadratic form

\[
h_0(u) = \int |\nabla u|^2 \, d\mu
\]

with the domain \(\text{D}(h_0) = W^{1,2}(E) \subset L^2(E)\). The quadratic form \(h_0\) is non-negative, densely defined (since \(C_\infty^0 (E) \subset \text{D}(h_0)\)) and closed (see Section 1.2).

2. By \(h_1\) we denote the quadratic form

\[
h_1(u) = \int \langle (\text{Re } V)^+ u, u \rangle + i \langle (\text{Im } V)u, u \rangle \, d\mu
\]

with the domain

\[
\text{D}(h_1) = \{ u \in L^2(E) : \int \left| \langle (\text{Re } V)^+ u, u \rangle + i \langle (\text{Im } V)u, u \rangle \right| \, d\mu < +\infty \}\.
\]

Here \(\langle \cdot, \cdot \rangle\) denotes the fiberwise inner product in \(E_x\).

In what follows, we will denote by \(h_1(\cdot, \cdot)\) the corresponding sesquilinear form obtained via polarization identity from \(h_1\).

The quadratic form \(h_1\) is sectorial. Indeed, by the inequalities

\[
|\langle (\text{Im } V)u(x), u(x) \rangle| \leq |(\text{Im } V)(x)u(x)||u(x)| \leq |(\text{Im } V)(x)||u(x)|^2
\]

and by (1.4), for all \(u \in \text{D}(h_1)\), the values of \(h_1(u)\) lie in a sector of \(\mathbb{C}\) with vertex \(\gamma = 0\). The form \(h_1\) is densely defined since, by (i) of Assumption A, we have
$C_c^\infty(E) \subset D(h_1)$. The form $h_1$ is closed. Indeed, by Theorem VI.1.11 in [6], it suffices to show that the pre-Hilbert space $D(h_1)$ with the inner product

$$(u,v)_{h_1} = (\text{Re} h_1)(u,v) + (u,v) = \int \langle (\text{Re} V)^+ u, v \rangle d\mu + (u,v),$$

is complete. Here $(\cdot, \cdot)$ denotes the inner product in $L^2(E)$ and $(\text{Re} h_1)(\cdot, \cdot)$ denotes the real part of the sesquilinear form $h_1(\cdot, \cdot)$ (see the definition below the equation (1.9) in Section VI.1.1 of [6]).

By (1.7), (1.8) and (1.4), it follows that $D(h_1)$ is the set of all $u \in L^2(E)$ such that $\|u\|^2_{h_1} < +\infty$, where $\| \cdot \|_{h_1}$ denotes the norm corresponding to the inner product $(\cdot, \cdot)_{h_1}$. By Example VI.1.15 in [6], it follows that $D(h_1)$ is complete.

3. By $h_2$ we denote the quadratic form

$$(1.9) \quad h_2(u) = \int \langle -(\text{Re} V)^- u, u \rangle d\mu$$

with the domain

$$(1.10) \quad D(h_2) = \{ u \in L^2(E) : \int \langle (\text{Re} V)^- u, u \rangle d\mu < +\infty \},$$

where $(\cdot, \cdot)$ denotes the fiberwise inner product in $E_x$.

The form $h_2$ is densely defined because, by (i) of Assumption A, we have $C_c^\infty(E) \subset D(h_2)$. Moreover, since for all $x \in M$, the operator $(\text{Re} V)^-(x) : E_x \to E_x$ is self-adjoint, it follows that the quadratic form $h_2$ is symmetric.

We make the following assumption on $h_2$.

**Assumption B.** Assume that $h_2$ is $h_0$-bounded with relative bound $b < 1$, i.e.

(i) $D(h_2) \supset D(h_0)$,

(ii) there exist constants $a \geq 0$ and $0 \leq b < 1$ such that

$$(1.11) \quad |h_2(u)| \leq a\|u\|^2 + b|h_0(u)|, \quad \text{for all } u \in D(h_0),$$

where $\| \cdot \|$ denotes the norm in $L^2(E)$.

**Remark 1.4.** If $(M, g)$ is a manifold of bounded geometry, Assumption B holds if $(\text{Re} V)^- \in L^p(\text{End} E)$, where $p = n/2$ for $n \geq 3$, $p > 1$ for $n = 2$, and $p = 1$ for $n = 1$. For the proof, see, for example, the proof of Remark 2.1 in [7].

**1.5 A realization of $H_V$ in $L^2(E)$.** We define an operator $S$ in $L^2(E)$ by the formula $Su = H_V u$ on the domain

$$(1.12) \quad \{ u \in W^{1,2}(E) : \int |\langle (\text{Re} V)^+ u, u \rangle + i\langle (\text{Im} V) u, u \rangle | \, d\mu < +\infty \text{ and } H_V u \in L^2(E) \}.$$
Remark 1.6. For all $u \in D(h_0) = W^{1,2}(E)$ we have $\nabla^* \nabla u \in W^{-1,2}(E)$. From Corollary 2.11 below it follows that for all $u \in W^{1,2}(E) \cap D(h_1)$, we have $V u \in L^1_{loc}(E)$. Thus $H_V u$ in (1.12) is a distributional section of $E$, and the condition $H_V u \in L^2(E)$ makes sense.

Remark 1.7. By (1.4) and by (1.8), the set $\text{Dom}(S)$ in (1.12) is equal to
\[ \{ u \in W^{1,2}(E) : \int \langle (\text{Re} V)^+ u, u \rangle \, d\mu < +\infty \text{ and } H_V u \in L^2(E) \}. \]

We now state the main result.

**Theorem 1.8.** Assume that $(M, g)$ is a manifold of bounded geometry with positive $C^\infty$-bounded measure $d\mu$, $E$ is a Hermitian vector bundle of bounded geometry over $M$, and $\nabla$ is a $C^\infty$-bounded Hermitian connection on $E$. Suppose that Assumptions A and B hold. Then the operator $S$ is $m$-sectorial.

Remark 1.9. Theorem 1.8 extends a result of T. Kato; see Theorem VI.4.6(a) in [6] (see also Remark 5(a) in [5]) which was proven for the operator $-\Delta + V$, where $\Delta$ is the standard Laplacian on $\mathbb{R}^n$ with the standard metric and measure, and $V \in L^1_{loc}(\mathbb{R}^n)$ is as in Assumptions A and B above (with $\text{Im} V = 0$). Theorem 1.8 also extends the result in [7] which establishes the self-adjointness of a realization in $L^2(E)$ of $H_V = \nabla^* \nabla + V$ on manifold $(M, g)$ with $d\mu$, $E$, and $\nabla$ as in the hypotheses of Theorem 1.8, and $V = V_1 + V_2$, where $0 \leq V_1 \in L^1_{loc}(\text{End} E)$ and $0 \geq V_2 \in L^1_{loc}(\text{End} E)$ are linear self-adjoint bundle endomorphisms satisfying Assumptions A and B (with $\text{Im} V = 0$).

2. Proof of Theorem 1.8

We adopt the arguments from Section VI.4.3 in [6] to our setting with the help of a more general version of Kato’s inequality (2.1).

2.1 Kato’s inequality. We begin with the following variant of Kato’s inequality for Bochner Laplacian (for the proof, see Theorem 5.7 in [2]).

**Lemma 2.2.** Assume that $(M, g)$ is a Riemannian manifold. Assume that $E$ is a Hermitian vector bundle over $M$ and $\nabla$ is a Hermitian connection on $E$. Assume that $w \in L^1_{loc}(E)$ and $\nabla^* \nabla w \in L^1_{loc}(E)$. Then
\[ (2.1) \quad \Delta_M |w| \leq \text{Re} \langle \nabla^* \nabla w, \text{sign} w \rangle, \]
where
\[ \text{sign} w(x) = \begin{cases} \frac{w(x)}{|w(x)|} & \text{if } w(x) \neq 0, \\ 0 & \text{otherwise}. \end{cases} \]

Remark 2.3. The original version of Kato’s inequality was proven in Kato [3].
2.4 Positivity. In what follows, we will use the following lemma whose proof is given in Appendix B of [2].

**Lemma 2.5.** Assume that \((M, g)\) is a manifold of bounded geometry with a smooth positive measure \(d\mu\). Assume that

\[
(b + \Delta_M)u = \nu \geq 0, \quad u \in L^2(M),
\]

where \(b > 0\), \(\Delta_M = d^*d\) is the scalar Laplacian on \(M\), and the inequality \(\nu \geq 0\) means that \(\nu\) is a positive distribution on \(M\), i.e. \((\nu, \phi) \geq 0\) for any \(0 \leq \phi \in C_c^\infty(M)\).

Then \(u \geq 0\) (almost everywhere or, equivalently, as a distribution).

**Remark 2.6.** It is not known whether Lemma 2.5 holds if \(M\) is an arbitrary complete Riemannian manifold. For more details about difficulties in the case of arbitrary complete Riemannian manifolds, see Appendix B of [2].

**Lemma 2.7.** The quadratic form \(h := (h_0 + h_1) + h_2\) is densely defined, sectorial, and closed.

**Proof:** Since \(h_0\) and \(h_1\) are sectorial and closed, it follows by Theorem VI.1.31 from [6] that \(h_0 + h_1\) is sectorial and closed. By (i) of Assumption B it follows that \(D(h_2) \supset D(h_0) \cap D(h_1)\), and by (1.11), (1.5), and (1.6), the following inequality holds:

\[
|h_2(u)| \leq a\|u\|^2 + b|h_0(u) + h_1(u)|, \quad \text{for all } u \in D(h_0) \cap D(h_1),
\]

where \(\|\cdot\|\) denotes the norm in \(L^2(E)\), and \(a \geq 0\) and \(0 \leq b < 1\) are as in (1.11).

Thus the quadratic form \(h_2\) is \((h_0 + h_1)\)-bounded with relative bound \(b < 1\). Since \(h_0 + h_1\) is a closed sectorial form, by Theorem VI.1.33 from [6], it follows that \(h = (h_0 + h_1) + h_2\) is a closed sectorial form. Since \(C_c^\infty(E) \subset D(h_0) \cap D(h_1) \subset D(h_2)\), it follows that \(h\) is densely defined. \(\square\)

In what follows, \(h(\cdot, \cdot)\) will denote the corresponding sesquilinear form obtained from \(h\) via polarization identity.

2.8 \textit{m-sectorial operator} \(H\) \textit{associated with} \(h\). Since \(h\) is a densely defined, closed and sectorial form in \(L^2(E)\), by Theorem VI.2.1 from [6] there exists an \(m\)-sectorial operator \(H\) in \(L^2(E)\) such that

(i) \(\text{Dom}(H) \subset D(h)\) and

\[h(u, v) = (Hu, v), \quad \text{for all } u \in \text{Dom}(H) \text{ and } v \in D(h),\]

(ii) \(\text{Dom}(H)\) is a core of \(h\),

(iii) if \(u \in \text{Dom}(H), \ w \in L^2(E),\) and

\[h(u, v) = (w, v)\]

holds for every \(v\) belonging to a core of \(h\), then \(u \in \text{Dom}(H)\) and \(Hu = w\).

The operator \(H\) is uniquely determined by condition (i).

We will also use the following lemma.
Lemma 2.9. Assume that $0 \leq T \in L^1_{\text{loc}}(\text{End} E)$ is a linear self-adjoint bundle map. Assume also that $u \in Q(T)$, where $Q(T) = \{ u \in L^2(E) : \langle Tu, u \rangle \in L^1(M) \}$. Then $Tu \in L^1_{\text{loc}}(E)$.

Proof: By adding a constant we can assume that $T \geq 1$ (in operator sense).

Assume that $u \in Q(T)$. We choose (in a measurable way) an orthogonal basis in each fiber $E_x$ and diagonalize $1 \leq T(x) \in \text{End}(E_x)$ to get $T(x) = \text{diag}(c_1(x), c_2(x), \ldots, c_m(x))$, where $0 < c_j \in L^1_{\text{loc}}(M)$, $j = 1, 2, \ldots, m$ and $m = \dim E_x$.

Let $u_j(x)$ ($j = 1, 2, \ldots, m$) be the components of $u(x) \in E_x$ with respect to the chosen orthogonal basis of $E_x$. Then for all $x \in M$

$$\langle Tu, u \rangle = \sum_{j=1}^{m} c_j(x)|u_j(x)|^2.$$

Since $u \in Q(T)$, we know that $0 < \int \langle Tu, u \rangle \, d\mu < +\infty$. Since $c_j > 0$, it follows that $c_j|u_j|^2 \in L^1(M)$, for all $j = 1, 2, \ldots, m$.

Now, for all $x \in M$ and $j = 1, 2, \ldots, m$

(2.2)\[ 2|c_ju_j| = 2|c_j||u_j| \leq |c_j| + |c_j||u_j|^2.\]

The right hand side of (2.2) is clearly in $L^1_{\text{loc}}(M)$. Therefore $c_ju_j \in L^1_{\text{loc}}(M)$.

But $(Tu)(x)$ has components $c_j(x)u_j(x)$ ($j = 1, 2, \ldots, m$) with respect to chosen bases of $E_x$. Therefore $Tu \in L^1_{\text{loc}}(E)$, and the lemma is proven.

Corollary 2.10. If $u \in D(h_1)$, then $((\text{Re} V)^+ + i(\text{Im} V))u \in L^1_{\text{loc}}(E)$.

Proof: Let $u \in D(h_1)$. Then $\langle (\text{Re} V)^+ u, u \rangle \in L^1(M)$, and, hence, by Lemma 2.9 we get $(\text{Re} V)^+ u \in L^1_{\text{loc}}(E)$. By (1.4) we obtain $|\text{Im} V||u|^2 \in L^1(M)$. Since for all $x \in M$ we have

$$2|\text{Im} V(x)u(x)| \leq 2|\text{Im} V(x)||u(x)| \leq |\text{Im} V(x)| + |\text{Im} V(x)||u(x)|^2,$$

and since, by Assumption A, $|\text{Im} V| \in L^1_{\text{loc}}(M)$, it follows that $(\text{Im} V)u \in L^1_{\text{loc}}(E)$, and the corollary is proven.

Corollary 2.11. If $u \in D(h)$, then $Vu \in L^1_{\text{loc}}(E)$.

Proof: Let $u \in D(h) = D(h_0) \cap D(h_1)$. By Corollary 2.10 it follows that $((\text{Re} V)^+ + i(\text{Im} V))u \in L^1_{\text{loc}}(E)$. Since $D(h) \subset D(h_2)$ and since $(\text{Re} V)^-(x) \geq 0$ as an operator $E_x \to E_x$, by Lemma 2.9 we have $(\text{Re} V)^- u \in L^1_{\text{loc}}(E)$. Thus $Vu \in L^1_{\text{loc}}(E)$, and the corollary is proven.
Lemma 2.12. The following operator relation holds: $H \subset S$.

Proof: We will show that for all $u \in \text{Dom}(H)$, we have $Hu = H_V u$.

Let $u \in \text{Dom}(H)$. By property (i) of Section 2.8 we have $u \in D(h)$; hence, by Corollary 2.11 we get $Vu \in L^1_{\text{loc}}(E)$. Then, for any $v \in C_c^\infty(E)$, we have

\begin{equation}
(Hu, v) = h(u, v) = (\nabla u, \nabla v) + \int \langle Vu, v \rangle \, d\mu,
\end{equation}

where $(\cdot, \cdot)$ denotes the $L^2$-inner product.

The first equality in (2.3) holds by property (i) from Section 2.8, and the second equality holds by definition of $h$.

Hence, using integration by parts in the first term on the right hand side of the second equality in (2.3) (see, for example Lemma 8.8 from [2]), we get

\begin{equation}
(u, \nabla^* \nabla v) = \int \langle Hu - Vu, v \rangle \, d\mu, \quad \text{for all } v \in C_c^\infty(E).
\end{equation}

Since $Vu \in L^1_{\text{loc}}(E)$ and $Hu \in L^2(E)$, it follows that $(Hu - Vu) \in L^1_{\text{loc}}(E)$, and (2.4) implies $\nabla^* \nabla u = Hu - Vu$ (as distributional sections of $E$). Therefore,

$$
\nabla^* \nabla u + Vu = Hu,
$$

and this shows that $Hu = H_V u$ for all $u \in \text{Dom}(H)$.

Now by definition of $S$ it follows that $\text{Dom}(H) \subset \text{Dom}(S)$ and $Hu = Su$ for all $u \in \text{Dom}(H)$. Therefore $H \subset S$, and the lemma is proven. \qed

Lemma 2.13. $C_c^\infty(E)$ is a core of the quadratic form $h_0 + h_1$.

Proof: It suffices to show (see Theorem VI.1.21 in [6] and the paragraph above the equation (1.31) in Section VI.1.3 of [6]) that $C_c^\infty(E)$ is dense in the Hilbert space $D(h_0 + h_1) = D(h_0) \cap D(h_1)$ with the inner product

$$
(u, v)_{h_0 + h_1} := h_0(u, v) + (\text{Re } h_1)(u, v) + (u, v),
$$

where $h_0(\cdot, \cdot)$ denotes the sesquilinear form corresponding to $h_0$ via polarization identity and $(\text{Re } h_1)$ denotes the real part of the sesquilinear form $h_1(\cdot, \cdot)$. Let $u \in D(h_0 + h_1)$ and $(u, v)_{h_0 + h_1} = 0$ for all $v \in C_c^\infty(E)$. We will show that $u = 0$.

We have

\begin{equation}
0 = h_0(u, v) + (\text{Re } h_1)(u, v) + (u, v) = (u, \nabla^* \nabla v) + \int \langle (\text{Re } V)^+ u, v \rangle \, d\mu + (u, v).
\end{equation}
Here we used integration by parts in the first term on the right hand side of the second equality.

Since \( u \in D(h_0 + h_1) \subset D(h_1) \), it follows that \( |\langle (\Re V)^+ u, u \rangle + i \langle (\Im V) u, u \rangle| \in L^1(M) \). Hence \( \langle (\Re V)^+ u, u \rangle \in L^1(M) \). By Lemma 2.9 we get \( (\Re V)^+ u \in L^1_{\text{loc}}(E) \). From (2.5) we get the following distributional equality:

\[
\nabla^* \nabla u = (- (\Re V)^+ - 1) u.
\]

From (2.6) we have \( \nabla^* \nabla u \in L^1_{\text{loc}}(E) \). By Lemma 2.2 and by (2.6), we obtain

\[
\Delta_M |u| \leq \Re \langle \nabla^* \nabla u, \text{sign} u \rangle = \langle -((\Re V)^+ + 1)|u|, \text{sign} u \rangle \leq -|u|.
\]

The last inequality in (2.7) holds since \( (\Re V)^+(x) \geq 0 \) as an operator \( E_x \to E_x \). Therefore,

\[
(\Delta_M + 1)|u| \leq 0.
\]

By Lemma 2.5, it follows that \( |u| \leq 0 \). So \( u = 0 \), and the lemma is proven.

**Lemma 2.14.** \( C_\infty^c(E) \) is a core of the quadratic form \( h = (h_0 + h_1) + h_2 \).

**Proof:** Since the quadratic form \( h_2 \) is \( (h_0 + h_1) \)-bounded, the lemma follows immediately from Lemma 2.13.

**3. Proof of Theorem 1.8**

By Lemma 2.12 we have \( H \subset S \), so it is enough to show that \( \text{Dom}(S) \subset \text{Dom}(H) \).

Let \( u \in \text{Dom}(S) \). By definition of \( \text{Dom}(S) \) in Section 1.5, we have \( u \in D(h_0) \subset D(h_2) \) and \( u \in D(h_1) \). Hence \( u \in D(h) \).

For all \( v \in C_\infty^c(E) \) we have

\[
h(u,v) = h_0(u,v) + h_1(u,v) + h_2(u,v) = (u, \nabla^* \nabla v) + \int \langle Vu, v \rangle d\mu = (H_V u, v).
\]

The last equality holds since \( H_V u = Su \in L^2(E) \). By Lemma 2.14 it follows that \( C_\infty^c(E) \) is a form core of \( h \). Now from property (iii) of Section 2.8 we have \( u \in \text{Dom}(H) \) with \( Hu = H_V u \). This concludes the proof of the theorem.

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