Subgroups and products of $\mathbb{R}$-factorizable $P$-groups

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Abstract. We show that every subgroup of an $\mathbb{R}$-factorizable abelian $P$-group is topologically isomorphic to a closed subgroup of another $\mathbb{R}$-factorizable abelian $P$-group. This implies that closed subgroups of $\mathbb{R}$-factorizable $P$-groups are not necessarily $\mathbb{R}$-factorizable. We also prove that if a Hausdorff space $Y$ of countable pseudocharacter is a continuous image of a product $X = \prod_{i \in I} X_i$ of $P$-spaces and the space $X$ is pseudo-$\omega_1$-compact, then $nw(Y) \leq \aleph_0$. In particular, direct products of $\mathbb{R}$-factorizable $P$-groups are $\mathbb{R}$-factorizable and $\omega$-stable.

Keywords: $P$-space, $P$-group, pseudo-$\omega_1$-compact, $\omega$-stable, $\mathbb{R}$-factorizable, $\aleph_0$-bounded, pseudocharacter, cellularity, $\aleph_0$-box topology, $\sigma$-product

Classification: Primary 54H11, 22A05, 54G10; Secondary 54A25, 54C10, 54C25

1. Introduction

The main subject of this article are $P$-groups, that is, topological groups in which all $G_\delta$-sets are open. It is known that $P$-groups are peculiar in many respects. For example, every $P$-group $G$ has a local base at the identity of open subgroups and if $G$ is $\aleph_0$-bounded, it has a local base at the identity of open normal subgroups [15, Lemma 2.1]. Weak compactness type conditions substantially improve the properties of $P$-groups. The following result proved in [15] demonstrates this phenomenon and will be frequently used in the article.

Theorem 1.1 ([15, Theorem 4.16 and Corollary 4.14]). For a $P$-group $G$, the following conditions are equivalent:

(1) $G$ is $\mathbb{R}$-factorizable;
(2) $G$ is pseudo-$\omega_1$-compact;
(3) $G$ is $\omega$-stable;
(4) $G$ is $\aleph_0$-bounded and every continuous homomorphic image $H$ of $G$ with $\psi(H) \leq \aleph_1$ is Lindelöf.

In addition, every $\mathbb{R}$-factorizable $P$-group $G$ satisfies $c(G) \leq \aleph_1$.

All terms that appear in Theorem 1.1 are explained in the next subsection.

Subgroups of $\mathbb{R}$-factorizable $P$-groups need not be $\mathbb{R}$-factorizable (see [13, Example 2.1] or [15, Example 3.28]). It is an open problem whether every $\aleph_0$-bounded $P$-group is topologically isomorphic to a subgroup of an $\mathbb{R}$-factorizable...
P-group (see Problem 4.1). We show, however, that every subgroup of an ℝ-factorizable abelian P-group can be embedded as a closed subgroup into another ℝ-factorizable abelian P-group (see Theorem 2.5). Hence closed subgroups of ℝ-factorizable P-groups can fail to be ℝ-factorizable. This is the main result of Section 2.

By [15, Theorem 5.5], direct products of ℝ-factorizable P-groups are ℝ-factorizable. In Theorem 3.7, we present a purely topological result about a special representation of continuous maps of products of P-spaces which generalizes Theorem 5.5 of [15]. It implies, in particular, that for any product of P-spaces, the properties of being ω-stable and pseudo-ω₁-compact are equivalent.

1.1 Notation and terminology. All spaces and topological groups are assumed to be Hausdorff unless a different axiom of separation is specified explicitly.

Let \( \{X_i : i \in I\} \) be a family of topological spaces. A subset \( B \) of the product \( X = \prod_{i \in I} X_i \) is called a box in \( X \) if it has the form \( B = \prod_{i \in I} B_i \), where \( B_i \subseteq X_i \) for each \( i \in I \). Given a box \( B \subseteq X \), we define the set \( \text{coord} B = \{ i \in I : B_i \neq X_i \} \).

The \( \aleph_0 \)-box topology of the product \( X \) is the topology generated by all boxes of the form \( U = \prod_{i \in I} U_i \), where \( |\text{coord} U| \leq \aleph_0 \) and each \( U_i \) is open in \( X_i \). Clearly, the Tychonoff topology of the space \( X \) is generated by open boxes \( U \) with \( |\text{coord} U| < \aleph_0 \).

For every nonempty set \( J \subseteq I \), we put \( X_J = \prod_{i \in J} X_i \) and denote by \( \pi_J \) the projection of \( X \) onto \( X_J \). Given a map \( f : X \rightarrow Y \), we say that \( f \) depends only on a set \( J \subseteq I \) if \( f(x) = f(y) \) for all \( x, y \in X \) satisfying \( \pi_J(x) = \pi_J(y) \).

Pick a point \( a \in X \) and, for every \( x \in X \), put

\[ \text{supp}(x) = \{ i \in I : x_i \neq a_i \}. \]

Then the subset

\[ \sigma(a) = \{ x \in X : \text{supp}(x) \text{ is finite} \} \]

of \( X \) is called the \( \sigma \)-product of the family \( \{X_i : i \in I\} \) with center at \( a \).

Let \( G = \prod_{i \in I} G_i \) be a direct product of groups. For every \( x \in G \), we set \( \text{supp} x = \{ i \in I : x_i \neq e_i \} \), where \( e_i \) is the identity of \( G_i \). Then the \( \sigma \)-product \( \sigma(e) \subseteq G \) is a subgroup of \( G \), where \( e \) is the identity of \( G \).

Suppose that \( Y \) is a space. We say that \( Y \) is a \( P \)-space if every countable intersection of open sets is open in \( Y \). Let \( \tau \) be an infinite cardinal. A subset \( Z \subseteq Y \) is said to be \( G_\tau \)-dense in \( Y \) if \( Z \) intersects every nonempty \( G_\tau \)-set in \( Y \).

A space \( Y \) is called \( \omega \)-stable if every continuous image \( Z \) of \( Y \) which admits a coarser second countable Tychonoff topology satisfies \( \text{nw}(Z) \leq \aleph_0 \). In general, let \( \tau \geq \aleph_0 \). A space \( Y \) is called \( \tau \)-stable if every continuous image \( Z \) of \( Y \) which admits a coarser Tychonoff topology of weight \( \leq \tau \) satisfies \( \text{nw}(Z) \leq \aleph_0 \). If \( Y \)
is $\tau$-stable for $\tau \geq \aleph_0$, then $Y$ is said to be stable. It is known that arbitrary products and $\sigma$-products of second countable spaces are $\omega$-stable [1, Corollary 13].

A space $Y$ is said to be *pseudo-$\omega_1$-compact* if every locally finite (equivalently, discrete) family of open sets in $Y$ is countable. Lindelöf spaces as well as spaces of countable cardinality are pseudo-$\omega_1$-compact.

A topological group $G$ is called $\aleph_0$-bounded if it can be covered by countably many translates of any neighborhood of the identity. We also say that $G$ is $\mathbb{R}$-factorizable if every continuous real-valued function $f$ on $G$ can be represented in the form $f = h \circ \varphi$, where $\varphi : G \to H$ is a continuous homomorphism onto a second countable topological group $H$ and $h$ is a continuous real-valued function on $H$. Every $\mathbb{R}$-factorizable group is $\aleph_0$-bounded, but not vice versa [13], [14].

The kernel of a homomorphism $p : G \to H$ is $\ker p$. The minimal subgroup of a group $G$ containing a set $A \subseteq G$ is denoted by $\langle A \rangle$.

As usual, $w(Y)$, $nw(Y)$, $\psi(Y)$, $L(Y)$, and $c(Y)$ are the weight, network weight, pseudocharacter, Lindelöf number and cellularity of a space $Y$, respectively.

The set of all positive integers is denoted by $\mathbb{N}$, while $\mathbb{Z}$ is the additive group of integers.

**2. Subgroups of $\mathbb{R}$-factorizable $P$-groups**

Here we show that an arbitrary subgroup of an $\mathbb{R}$-factorizable abelian $P$-group is topologically isomorphic to a closed subgroup of another $\mathbb{R}$-factorizable abelian $P$-group. This result enables us to conclude that closed subgroups of $\mathbb{R}$-factorizable $P$-groups are not necessarily $\mathbb{R}$-factorizable. Since, by Theorem 1.1, $\mathbb{R}$-factorizability and pseudo-$\omega_1$-compactness coincide for $P$-groups, this makes $\mathbb{R}$-factorizable $P$-groups look like pseudocompact groups: every subgroup of a pseudocompact group is topologically isomorphic to a closed subgroup of another pseudocompact group [4]. This analogy between $\mathbb{R}$-factorizable $P$-groups and pseudocompact groups will be extended in Section 3.

We start with several auxiliary facts.

**Lemma 2.1.** Suppose that $G$ is an $\mathbb{R}$-factorizable $P$-group, and let $H$ be a $G_{\omega_1}$-dense subgroup of $G$. Then $H$ is $\mathbb{R}$-factorizable.

**Proof:** By Theorem 1.1, $G$ satisfies $c(G) \leq \aleph_1$. Therefore, the dense subgroup $H$ of $G$ also satisfies $c(H) \leq \aleph_1$. Let $f : H \to \mathbb{R}$ be a continuous function. By Schepin’s theorem in [12], one can find a quotient homomorphism $\pi : H \to K$ onto a topological group $K$ with $\psi(K) \leq \aleph_1$ and a continuous function $g : K \to \mathbb{R}$ such that $f = g \circ \pi$. Observe that $H \subseteq G \subseteq \varrho G = \varrho H$, where $\varrho G$ and $\varrho H$ denote the Raïkov completions of $G$ and $H$, respectively. Now, consider the continuous homomorphic extension $\hat{\pi} : \varrho H \to \varrho K$ of $\pi$, and take the restriction $\hat{\pi} = \hat{\pi} \upharpoonright G : G \to \varrho K$ of $\hat{\pi}$ to $G$. Since $H$ is $G_{\omega_1}$-dense in $G$, the image $K = \hat{\pi}(H)$ is $G_{\omega_1}$-dense in $\hat{\pi}(G)$. We claim that $\hat{\pi}(G) = K$. 


Indeed, $\psi(K) \leq \aleph_1$ implies that there exists a family \( \{U_\alpha : \alpha < \omega_1\} \) of open sets in \( \tilde{\pi}(G) \) such that \( \{e\} = K \cap \bigcap_{\alpha \in \omega_1} U_\alpha \), where \( e \) is the identity of \( gK \). If \( P = \bigcap_{\alpha \in \omega_1} U_\alpha \setminus \{e\} \neq \emptyset \), then \( P \) is a nonempty \( G_{\omega_1} \)-set in \( \tilde{\pi}(G) \) that does not intersect \( K \), which is a contradiction. Thus, \( \psi(\tilde{\pi}(G)) \leq \aleph_1 \). Since every fiber of \( \tilde{\pi} \) is a \( G_{\omega_1} \)-set in \( G \), the group \( H \) intersects all fibers of \( \tilde{\pi} \). Hence we have \( \tilde{\pi}(G) = \tilde{\pi}(H) = K \). So, \( \tilde{f} = g \circ \tilde{\pi} \) is a continuous extension of \( f \) to \( G \). This implies that \( H \) is \( C \)-embedded in \( G \) and, hence, \( H \) is \( \mathbb{R} \)-factorizable by [7, Theorem 2.4]. □

Pseudo-\( \omega_1 \)-compactness is not a productive property, not even in the class of \( P \)-spaces (one can modify Novák’s construction in [11] to produce a counterexample). The following lemma shows the difference between \( P \)-spaces and \( P \)-groups.

**Lemma 2.2.** A finite product of \( \mathbb{R} \)-factorizable \( P \)-groups is pseudo-\( \omega_1 \)-compact (equivalently, \( \mathbb{R} \)-factorizable).

**Proof:** Let \( G = G_1 \times \cdots \times G_n \), where each \( G_i \) is an \( \mathbb{R} \)-factorizable \( P \)-group. Then \( G \) is also a \( P \)-group. Hence we can assume that \( n = 2 \). Note that the factors \( G_1 \) and \( G_2 \) are \( \aleph_0 \)-bounded, and so is the product group \( G \). So, by Theorem 1.1, it suffices to verify that every continuous homomorphic image \( H \) of \( G \) with \( \psi(H) \leq \aleph_1 \) is Lindelöf. Let \( p : G \to H \) be a corresponding homomorphism. Then one can apply [14, Lemma 3.7] to find, for every \( i = 1, 2 \), a continuous homomorphism \( f_i : G_i \to K_i \) onto a topological group \( K_i \) with \( \psi(K_i) \leq \aleph_1 \) such that \( \ker f_1 \times \ker f_2 \subseteq \ker p \). Refining topologies of the groups \( K_i \), we can assume that the homomorphisms \( f_1 \) and \( f_2 \) are open. Then \( K_1 \) and \( K_2 \) are \( P \)-groups by [15, Lemma 2.1] and the product homomorphism \( f = f_1 \times f_2 \) of \( G \) onto \( K = K_1 \times K_2 \) is open. From our choice of the homomorphisms \( f_1 \) and \( f_2 \) it follows that there exists a homomorphism \( \varphi : K \to H \) such that \( p = \varphi \circ f \). Since \( f \) is open, the homomorphism \( \varphi \) is continuous. By Theorem 1.1, the \( P \)-groups \( K_1 \) and \( K_2 \) are Lindelöf, and so is the product group \( K \) by Noble’s theorem in [10]. Hence the group \( H = \varphi(K) \) is Lindelöf as well. This finishes the proof. □

The next result has several applications in this section and in Section 3.

**Lemma 2.3.** The following conditions are equivalent for a product space \( X = \prod_{i \in I} X_i \):

(a) \( X \) is pseudo-\( \omega_1 \)-compact;
(b) the product \( X_J = \prod_{i \in J} X_i \) is pseudo-\( \omega_1 \)-compact for each finite set \( J \subseteq I \);
(c) every \( \sigma \)-product \( \sigma(a) \subseteq X \) is pseudo-\( \omega_1 \)-compact;
(d) every \( \sigma \)-product \( \sigma(a) \subseteq X \) endowed with the relative \( \aleph_0 \)-box topology is pseudo-\( \omega_1 \)-compact.

**Proof:** It clear that (a) \( \Rightarrow \) (b). Since, for each \( a \in X \), \( \sigma(a) \) is dense in \( X \) when \( X \) carries the usual product topology and the \( \aleph_0 \)-box topology is finer than the
product topology of $X$, we have that (c) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) $\Rightarrow$ (b). Therefore, it suffices to show that (b) $\Rightarrow$ (d).

Let $\{U_\alpha : \alpha < \omega_1\}$ be a collection of nonempty open sets in $\sigma(a)$. We shall show that this family cannot be discrete. Without loss of generality, we may assume that $U_\alpha = \sigma \cap V_\alpha$ for each $\alpha < \omega_1$, where $V_\alpha$ has the form $\prod_{i \in I} V_{\alpha,i}$, the sets $V_{\alpha,i}$ are open in $X_i$ and $\text{coord} V_\alpha \leq \aleph_0$. Take a point $x_\alpha \in U_\alpha$. Since $x_\alpha \in \sigma(a)$, the point $a(i) \in X_i$ is an element of $V_{\alpha,i}$ for all $i \in I \setminus J_\alpha$, where $J_\alpha = \text{supp}(x_\alpha)$ is a finite subset of $I$. Now we apply the $\Delta$-lemma in order to find a subset $A$ of $\omega_1$ of cardinality $\aleph_1$ and a finite set $J \subseteq I$ such that $J_\alpha \cap J_\beta = J$ whenever $\alpha, \beta \in A$ and $J_\alpha \neq J_\beta$. Since the space $X_J = \prod_{i \in J} X_i$ is pseudo-$\omega_1$-compact, there exists a point $y \in X_J$ such that every neighborhood of $y$ intersects infinitely many elements of the family $\{\prod_{i \in J} V_{\alpha,i} : \alpha \in A\}$. Define a point $x \in \sigma(a)$ by

$$x(i) = \begin{cases} y(i) & \text{if } i \in J; \\ a(i) & \text{if } i \in I \setminus J. \end{cases}$$

It is easy to see that $\pi_J(x) = y$ and every neighborhood of $x$ intersects an infinite number of elements of $\{U_\alpha : \alpha \in A\}$. Hence the space $\sigma(a)$ is pseudo-$\omega_1$-compact.

The equivalence of (a) and (b) in the above lemma should be a known result, but the authors have not found a corresponding reference in the literature.

**Corollary 2.4.** Let $\Pi = \prod_{i \in I} G_i$ be a direct product of $\mathbb{R}$-factorizable $P$-groups. Then $\sigma(e) \subseteq \Pi$, endowed with the relative $\aleph_0$-box topology, is an $\mathbb{R}$-factorizable $P$-group.

**Proof:** It is clear that $\sigma(e)$ is a $P$-group. Therefore, $\sigma(e)$ is $\mathbb{R}$-factorizable by Theorem 1.1, Lemma 2.2 and Lemma 2.3.

We now have all necessary tools to deduce the main result of this section about closed embeddings into $\mathbb{R}$-factorizable $P$-groups.

**Theorem 2.5.** Suppose that $G$ is an $\mathbb{R}$-factorizable abelian $P$-group. If $H$ is an arbitrary subgroup of $G$, then $H$ can be embedded as a closed subgroup into another $\mathbb{R}$-factorizable abelian $P$-group.

**Proof:** Let $\mathbb{Z}$ be the discrete group of integers. Clearly, $G \times \mathbb{Z}$ is an $\mathbb{R}$-factorizable abelian $P$-group that contains an isomorphic copy of $G$. Replacing $G$ by $G \times \mathbb{Z}$, if necessary, we may assume that $G$ contains an element $g$ of infinite order, $g \neq 0_G$.

Let $\lambda = |G| \cdot \aleph_2$ and put $\kappa = \lambda$ if $\lambda$ is a regular cardinal or $\kappa = \lambda^+$, otherwise. Consider the group

$$\sigma = \{x \in G^\kappa : |\text{supp } x| < \aleph_0\}$$

endowed with the relative $\aleph_0$-box topology inherited from $G^\kappa$. Then $\sigma$ is an $\mathbb{R}$-factorizable abelian $P$-group by Corollary 2.4 and, clearly, $|\sigma| = \kappa$. Let $\sigma \setminus \{0_\sigma\} = \{x \in \sigma : x \neq 0_\sigma\}$. Then $\sigma \setminus \{0_\sigma\}$ is an $\mathbb{R}$-factorizable abelian $P$-group such that $\sigma \setminus \{0_\sigma\}$ is a closed subgroup of $G$. Therefore, $H$ can be embedded as a closed subgroup into another $\mathbb{R}$-factorizable abelian $P$-group.

□
\{x_\alpha : \alpha < \kappa\}. To every element \(x_\alpha\), we assign an element \(\tilde{x}_\alpha \in \sigma\) recursively as follows. Choose \(\delta_0 > \max \sup \sup x_0\) and define \(\tilde{x}_0 \in \sigma\) by

\[
\tilde{x}_0(\nu) = \begin{cases} 
  x_0(\nu) & \text{if } \nu \neq \delta_0; \\
  g & \text{if } \nu = \delta_0.
\end{cases}
\]

Suppose that we have already defined \(\tilde{x}_\beta\) for each \(\beta < \alpha\), where \(\alpha < \kappa\). Choose \(\delta_\alpha > \sup(\sup x_\alpha \cup \bigcup_{\beta < \alpha} \sup \tilde{x}_\beta)\) and define a point \(\tilde{x}_\alpha \in \sigma\) by

\[
\tilde{x}_\alpha(\nu) = \begin{cases} 
  x_\alpha(\nu) & \text{if } \nu \neq \delta_\alpha; \\
  g & \text{if } \nu = \delta_\alpha.
\end{cases}
\]

It is clear that \(\delta_\alpha = \max \sup \tilde{x}_\alpha\). This finishes our construction.

Observe that the sequence \(\{\delta_\alpha : \alpha < \kappa\}\) is strictly increasing (hence it is cofinal in \(\kappa\)) and \(\tilde{x}_\beta(\delta_\alpha) = 0_G\) whenever \(\beta < \alpha < \kappa\). Consider the subgroup \(G_0 = \langle H_0 \cup B \rangle\) of \(\sigma\), where

\[H_0 = \{x \in \sigma : x(0) \in H \text{ and } x(\nu) = 0_G \text{ for each } \nu \neq 0\}\]

and \(B = \{\tilde{x}_\alpha : \alpha < \kappa\}\). We claim that the group \(G_0\) is \(\mathbb{R}\)-factorizable and contains \(H_0 \simeq H\) as a closed subgroup. It is easy to see that \(H_0\) is closed in \(G_0\) because it can be expressed as the intersection of the coordinate 0 axes with \(G_0\). Indeed, suppose that \(x \in G_0\) and \(x(\nu) = 0_G\) for all \(\nu > 0\). By the definition of \(G_0\), \(x\) has the form \(x = h + k_1 \tilde{x}_{\alpha_1} + \cdots + k_n \tilde{x}_{\alpha_n}\), where \(h \in H_0\), \(\alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa\) and \(k_i \in \mathbb{Z}\) for \(i = 1, \ldots, n\). Then \(\tilde{x}_{\alpha_i}(\delta_{\alpha_i}) = 0_G\) for each \(i < n\) and \(\tilde{x}_{\alpha_n}(\delta_{\alpha_n}) = g\). Hence \(k_n = 0\). If we proceed in the same way for \(i = n - 1, \ldots, 1\), we obtain \(k_n = \cdots = k_1 = 0\), whence \(x = h\), with \(h \in H_0\).

By Lemma 2.1, to prove that \(G_0\) is \(\mathbb{R}\)-factorizable, it suffices to verify that \(G_0\) is \(G_{\omega_1}\)-dense in \(\sigma\). To this end, it is enough to show that if \(x \in \sigma\), \(C \subseteq \kappa\) and \(|C| \leq \aleph_1\), then there exists \(\alpha < \kappa\) such that \(\tilde{x}_\alpha(\nu) = x(\nu)\) for each \(\nu \in C\). Suppose that \(x \in \sigma\) and choose \(\beta < \kappa\) such that \(\delta_\beta \sup C\). Then choose \(\alpha < \kappa\) such that \(\beta \leq \alpha\) and \(x_\alpha(\nu) = x(\nu)\) for each \(\nu < \delta_\beta\). Then \(\tilde{x}_\alpha(\nu) = x(\nu)\) for each \(\nu \in C\). This implies that the group \(G_0\) is \(G_{\omega_1}\)-dense in \(\sigma\) and, therefore, \(\mathbb{R}\)-factorizable.

\[\square\]

**Corollary 2.6.** Closed subgroups of \(\mathbb{R}\)-factorizable \(P\)-groups need not be \(\mathbb{R}\)-factorizable.

**Proof:** According to [13, Example 3.1], there exist an \(\mathbb{R}\)-factorizable abelian \(P\)-group \(G\) and a dense subgroup \(H\) of \(G\) such that \(H\) is not \(\mathbb{R}\)-factorizable. By Theorem 2.5, \(H\) is topologically isomorphic to a closed subgroup of another \(\mathbb{R}\)-factorizable \(P\)-group, so that closed subgroups of \(\mathbb{R}\)-factorizable \(P\)-groups are not necessarily \(\mathbb{R}\)-factorizable.

\[\square\]

It is known that all subgroups of compact groups as well as all subgroups of \(\sigma\)-compact groups are \(\mathbb{R}\)-factorizable [13], [14]. In the following definition, we introduce the class of groups with this property.
Definition 2.7. A topological group $G$ is called hereditarily $\mathbb{R}$-factorizable if all subgroups of $G$ are $\mathbb{R}$-factorizable.

Theorem 2.8. Every hereditarily $\mathbb{R}$-factorizable $P$-group is countable and, therefore, discrete.

Proof: Suppose to the contrary that $G$ is an uncountable hereditarily $\mathbb{R}$-factorizable $P$-group and take a subset $A$ of $G$ of cardinality $\aleph_1$. It is clear that the $P$-group $H = \langle A \rangle$ has cardinality $\aleph_1$. Since $H$ is $\mathbb{R}$-factorizable and $L(H) \leq \aleph_1$, from [15, Corollary 3.34] it follows that $H$ is a Lindelöf group. In its turn, this implies that $w(H) \leq \aleph_1$ (see [15, Corollary 4.11]). If $w(H) = \aleph_1$, then by [7, Theorem 3.1], $H$ has a subgroup which fails to be $\mathbb{R}$-factorizable, thus contradicting the hereditary $\mathbb{R}$-factorizability of $G$. Hence, $w(H) = \aleph_0$. Since $H$ is a $P$-space, it is discrete and, consequently, $|H| = w(H) = \aleph_0$. This contradiction completes the proof. \(\Box\)

One can reformulate Theorem 2.8 by saying that every uncountable $P$-group $G$ contains a subgroup of size $\aleph_1$ which fails to be $\mathbb{R}$-factorizable. Indeed, if $G$ is $\mathbb{R}$-factorizable, this immediately follows from the above argument. Otherwise, by Theorem 1.1, $G$ contains a discrete family $\{U_\alpha : \alpha < \omega_1\}$ of nonempty open sets. Choose a subgroup $H$ of $G$ of size $\aleph_1$ such that $V_\alpha = H \cap U_\alpha \neq \emptyset$ for each $\alpha < \omega_1$. Then the family $\{V_\alpha : \alpha < \omega_1\}$ of nonempty open sets is discrete in $H$, so that the group $H$ is not $\mathbb{R}$-factorizable by Theorem 1.1.

3. Continuous images

By [15, Theorem 5.5], an arbitrary direct product $G$ of $\mathbb{R}$-factorizable $P$-groups is $\mathbb{R}$-factorizable. Here we strengthen this result and show that every continuous map $f: G \to X$ to a Hausdorff space $X$ of countable pseudocharacter can be factored via a quotient homomorphism $\pi: G \to K$ onto a second countable topological group $K$. In fact, this follows from an even stronger result (see Theorem 3.7): if a Hausdorff space $Y$ of countable pseudocharacter is a continuous image of a product $X$ of $P$-spaces and $X$ is pseudo-$\omega_1$-compact, then $nw(Y) \leq \aleph_0$. In particular, the space $X$ is $\omega$-stable. We precede this result by a series of lemmas. The first of them is an analogue of Noble’s theorem on $z$-closed projections [9], [10].

Lemma 3.1. The Cartesian product $X \times Y$ of regular $P$-spaces $X$ and $Y$ is pseudo-$\omega_1$-compact if and only if $X$ and $Y$ are pseudo-$\omega_1$-compact and the projection $p: X \times Y \to X$ transforms clopen subsets of $X \times Y$ to clopen subsets of $X$.

Proof: Suppose that $X \times Y$ is pseudo-$\omega_1$-compact and let $W \subseteq X \times Y$ be a clopen set. If there exists a point $x_0 \in p(W) \setminus p(W)$, take any point $y_0 \in Y$ and a neighborhood $W'_0 = U'_0 \times V_0$ of $(x_0, y_0)$, where $U'_0$ and $V_0$ are clopen sets, such that $W'_0 \cap W = \emptyset$. Pick a point $(x_1, y_1) \in W$ with $x_1 \in U'_0$. Now we take neighborhoods $W_1 = U_1 \times V_1$ and $W'_1 = U'_1 \times V_1$ of $(x_1, y_1)$ and $(x_0, y_1)$, respectively, where $U_1$, \(\Box\)}
Lemma 3.2. This contradiction proves the lemma. □

**Proof:** Set \( V \subseteq W \) for each \( \beta < \alpha \), such that \( W_\beta = U_\beta \times V_\beta \) is a neighborhood of \((x_\beta, y_\beta)\) satisfying \( W_\beta \subseteq W \) and \( W_\beta = U_\beta \times V_\beta \) is a neighborhood of \((x_0, y_\beta)\) with \( W_\beta \cap W = \emptyset \), and where \( U_\beta \cup U'_\beta \subseteq U_\gamma \) if \( \gamma < \beta < \alpha \). Choose \((x_\alpha, y_\alpha)\) in \( W \) such that \( x_\alpha \in \bigcap_{\beta < \alpha} U'_\beta \). Then we can take neighborhoods \( W_\alpha = U_\alpha \times V_\alpha \) and \( W'_\alpha = U'_\alpha \times V_\alpha \) of \((x_\alpha, y_\alpha)\) and \((x_0, y_\alpha)\), respectively, such that \( W_\alpha \cap W = \emptyset \) and \( W_\alpha \subseteq W \), and where \( U_\alpha \cup U'_\alpha \subseteq \bigcap_{\beta < \alpha} U'_\beta \). This finishes our recursive construction.

Since \( X \times Y \) is pseudo-\( \omega_1 \)-compact, the family \( F = \{ W_\alpha : \alpha < \omega_1 \} \) has an accumulation point \((x, y)\) in \( W \). We claim that \((x, y)\) is an accumulation point of the family \( F' = \{ W'_\alpha : \alpha < \omega_1 \} \). Indeed, let \( \alpha_0 < \omega_1 \) be arbitrary. Since \( U_\alpha \cup U'_\alpha \subseteq U_\beta \) if \( \beta < \alpha < \omega_1 \) and each \( U'_\alpha \) is clopen, we have \( x \in \bigcap_{\alpha < \omega_1} U'_\alpha \). Let \( U \times V \) be a neighborhood of \((x, y)\) in \( X \times Y \). Since \( y \) is an accumulation point of the family \( \{ V_\alpha : \alpha < \omega_1 \} \), there exists \( \alpha > \alpha_0 \) such that \( V \cap V_\alpha \neq \emptyset \). Clearly, \( x \in U \cap U'_\alpha \), so that \( (U \times V) \cap (U'_\alpha \times V_\alpha) \neq \emptyset \). Our claim is proved.

Thus, \((x, y)\) \( \in \bigcup F \cap \bigcup F' \neq \emptyset \). However, \( \bigcup F \subseteq W \) and \( \bigcup F' \subseteq (X \times Y) \setminus W = W' \), whence \( \bigcup F \cap \bigcup F' \subseteq W \cap W' = \emptyset \). This contradiction shows that the set \( p(W) \) is clopen in \( X \).

Conversely, suppose that both spaces \( X \) and \( Y \) are pseudo-\( \omega_1 \)-compact and \( p : X \times Y \to X \) transforms clopen subsets of \( X \times Y \) to clopen subsets of \( X \). Suppose to the contrary that \( X \times Y \) contains a discrete family \( \{ O_\alpha : \alpha < \omega_1 \} \) of nonempty clopen sets. For every \( \alpha < \omega_1 \), put \( W_\alpha = \bigcup_{\beta \geq \alpha} O_\beta \). Then we have a decreasing sequence \( W_0 \supseteq W_1 \supseteq \cdots \supseteq W_\alpha \supseteq \cdots, \alpha < \omega_1 \), of nonempty clopen subsets of \( X \times Y \) with empty intersection. Each set \( U_\alpha = p(W_\alpha) \) is clopen in \( X \) and, since \( X \) is pseudo-\( \omega_1 \)-compact, the set \( \bigcap_{\alpha < \omega_1} U_\alpha \) is nonempty. Let \( x_0 \) be an element of \( \bigcap_{\alpha < \omega_1} U_\alpha \). The sets \( V_\alpha = (\{ x_0 \} \times Y) \cap W_\alpha \) are clopen in the pseudo-\( \omega_1 \)-compact space \( \{ x_0 \} \times Y \). Hence \( \bigcap_{\alpha < \omega_1} V_\alpha \subseteq \bigcap_{\alpha < \omega_1} W_\alpha \) is nonempty. This contradiction proves the lemma. □

**Lemma 3.2.** Suppose that the product \( X \times Y \) of \( P \)-spaces \( X \) and \( Y \) is pseudo-\( \omega_1 \)-compact. If \( W \) is a clopen set in \( X \times Y \), then for every \( x_0 \in p(W) \), there exists a clopen neighborhood \( U \) of \( x_0 \) in \( X \) such that \( U \times V_{x_0} \subseteq W \), where \( V_{x_0} = \{ y \in Y : (x_0, y) \in W \} \).

**Proof:** Set \( O = (X \times V_{x_0}) \setminus W \). Since \( V_{x_0} \) is clopen in \( Y \), the set \( O \) is clopen in \( X \times Y \). From Lemma 3.1 it follows that \( p(O) \) and \( U = X \setminus p(O) \) are clopen sets in \( X \), where \( p : X \times Y \to X \) is the projection. Note that \( x_0 \in U \) and if \((x, y) \in U \times V_{x_0} \), then \( x \notin p(O) \). So, \((x, y) \in W \) and, hence, \( U \times V_{x_0} \subseteq W \). □

The next result can be obtained by combining [8, Theorem 1.6] and the characterization of the so-called *approximation property* for products of two spaces given in [2]. We prefer, however, to supply the reader with a direct proof.
Lemma 3.3. Suppose that the product \( X = \prod_{i=1}^{k} X_i \) of \( P \)-spaces is pseudo-\( \omega_1 \)-compact. If \( W \) is a clopen set in \( X \), then \( W = \bigcup_{n \in \omega} \prod_{i=1}^{k} U_{n,i} \), where the sets \( U_{n,i} \) are clopen in \( X_i \) for all \( n \in \omega \) and \( i \leq k \).

**Proof:** By Lemma 3.1, it suffices to consider the case \( n = 2 \). Let \( W \) be a clopen subset of \( X_1 \times X_2 \). Then \( W' = X \setminus W \) is clopen as well. For every \( x \in X_1 \), put

\[
V_x = \{ y \in X_2 : (x,y) \in W \} \quad \text{and} \quad V'_x = \{ y \in X_2 : (x,y) \in W' \}.
\]

Then both sets \( V_x \) and \( V'_x \) are clopen in \( X_2 \) and \( V'_x = X_2 \setminus V_x \). Consider the equivalence relation \( \sim \) on \( X_1 \) defined by \( x \sim y \) if and only if \( V_x = V_y \). We claim that for every \( x \in X_1 \), the equivalence class \([x]\) of \( x \) is open in \( X_1 \). Indeed, if \( y \in [x] \), then \( V_y = V_x = V \). Apply Lemma 3.2 to choose a clopen neighborhood \( U \) of \( y \) in \( X_1 \) such that \( U \times V \subseteq W \) and \( U \times V' \subseteq W' \), where \( V' = X_2 \setminus V \). Then \( V_z = V \) for each \( z \in U \), so that \( y \in U \subseteq [x] \). This proves that the set \([x]\) is open.

Since the space \( X_1 \) is pseudo-\( \omega_1 \)-compact and the equivalence classes \([x]\) with \( x \in X_1 \) form a disjoint open cover of \( X_1 \), there exists a countable set \( \{ x_n : n \in \omega \} \subseteq X_1 \) such that \( X_1 = \bigcup_{n \in \omega} [x_n] \). It is clear that every set \( U_{n,1} = [x_n] \) is clopen in \( X_1 \). Therefore, \( W = \bigcup_{n \in \omega} U_{n,1} \times X_2 \) is the required representation of \( W \), where \( U_{n,2} = V_{x_n} \) for each \( n \in \omega \). \( \square \)

It is well known (see [6]) that if a product space \( X = \prod_{i \in I} X_i \) has countable cellularity, then every regular closed set in \( X \) depends on at most countably many coordinates. In a sense, our next result is an analogue of this fact in the case when the product space \( X \) is pseudo-\( \omega_1 \)-compact and the factors \( X_i \) are \( P \)-spaces.

Lemma 3.4. Suppose that a product \( X = \prod_{i \in I} X_i \) of \( P \)-spaces is pseudo-\( \omega_1 \)-compact. Let \( \sigma(a) \subseteq X \) be a \( \sigma \)-product endowed with the relative \( \aleph_0 \)-box topology (finer than the usual subspace topology). Then every clopen subset of \( \sigma(a) \) depends on at most countably many coordinates.

**Proof:** It is clear that the space \( \sigma(a) \) with the \( \aleph_0 \)-box topology is a \( P \)-space. Let \( U \) be a clopen subset of \( \sigma(a) \). Then \( V = \sigma(a) \setminus U \) is also clopen in \( \sigma(a) \). Suppose that \( \pi_j(U) \cap \pi_j(V) \neq \emptyset \) for every countable set \( J \subseteq I \). Let us call a set \( A \subseteq \sigma(a) \) canonical if \( A \) has the form \( \sigma(a) \cap P \), where \( P \) is an \( \aleph_0 \)-box in \( X \). First, we prove the following auxiliary fact.

**Claim.** Let \( A \subseteq U \) and \( B \subseteq V \) be canonical open sets in \( \sigma(a) \) such that \( U' = U \setminus A \neq \emptyset \) and \( V' = V \setminus B \neq \emptyset \). Then \( \pi_j(U') \cap \pi_j(V') \neq \emptyset \) for each countable set \( J \subseteq I \).

Indeed, there exists a nonempty countable set \( C \subseteq I \) such that \( A = \sigma(a) \cap \pi_C^{-1} \pi_C(A) \) and \( B = \sigma(a) \cap \pi_C^{-1} \pi_C(B) \). Let \( J \) be a countable subset of \( I \). We can assume that \( C \subseteq J \). Since \( A \cap V = \emptyset = B \cap U \), we infer that

\[
\pi_j(A) \cap \pi_j(V) = \emptyset \quad \text{and} \quad \pi_j(B) \cap \pi_j(U) = \emptyset.
\]
Note that the set $U' \cup A$ is dense in $U$ and $V' \cup B$ is dense in $V$. Since the restriction of $\pi_J$ to $\sigma(a)$ is an open map, from $\pi_J(U) \cap \pi_J(V) \neq \emptyset$ it follows that

$$
\pi_J(U' \cup A) \cap \pi_J(V' \cup B) \neq \emptyset.
$$

Note that $U' \subseteq U$ and $V' \subseteq V$, so (1) implies that $\pi_J(U') \cap \pi_J(B) = \emptyset$, $\pi_J(V') \cap \pi_J(A) = \emptyset$ and $\pi_J(A) \cap \pi_J(B) = \emptyset$. Therefore, from (2) it follows that $\pi_J(U') \cap \pi_J(V') \neq \emptyset$. This proves our claim.

We will construct by recursion three sequences $\{I_\alpha : \alpha < \omega_1\}$, $\{U_\alpha : \alpha < \omega_1\}$ and $\{V_\alpha : \alpha < \omega_1\}$ satisfying the following conditions for all $\beta, \gamma < \omega_1$:

(i) $I_\beta \subseteq I$, $|I_\beta| \leq \aleph_0$;
(ii) $I_\gamma \subseteq I_\beta$ if $\gamma < \beta$;
(iii) $U_\beta$ and $V_\beta$ are nonempty canonical clopen sets in $\sigma(a)$;
(iv) $U_\beta \subseteq U$, $V_\beta \subseteq V$ and $\pi_I(\beta) = \pi_I(\beta)$;
(v) $U_\gamma = \sigma(a) \cap \pi_I^{-1}(U_\gamma)$ and $V_\gamma = \sigma(a) \cap \pi_I^{-1}(V_\gamma)$ if $\gamma < \beta$;
(vi) $U_\gamma \cap U_\beta = \emptyset$ and $V_\gamma \cap V_\beta = \emptyset$ if $\gamma < \beta$.

To start, take a nonempty countable set $I_0 \subseteq I$ and choose canonical clopen sets $U_0$ and $V_0$ in $\sigma(a)$ such that $U_0 \subseteq U$, $V_0 \subseteq V$ and $\pi_I(0) \cap \pi_I(0) \neq \emptyset$. Taking smaller clopen sets, one can assume that $\pi_I(0) = \pi_I(0)$. Suppose that at some stage $\alpha < \omega_1$, we have defined sequences $\{I_\beta : \beta < \alpha\}$, $\{U_\beta : \beta < \alpha\}$ and $\{V_\beta : \beta < \alpha\}$ satisfying conditions (i)–(vi). Since each $I_\beta$ is countable and the sets $U_\beta$, $V_\beta$ depend on countably many coordinates, there exists a countable set $I_\alpha \subseteq I$ such that $I_\beta \subseteq I_\alpha$, $U_\beta = \sigma(a) \cap \pi_I^{-1}(U_\beta)$ and $V_\beta = \sigma(a) \cap \pi_I^{-1}(V_\beta)$ for each $\beta < \alpha$. Let $U'_\alpha = U\setminus U_\alpha$ and $V'_\alpha = V\setminus V_\alpha$, where $G_\alpha = \bigcup_{\beta < \alpha} U_\beta$ and $H_\alpha = \bigcup_{\beta < \alpha} V_\beta$. Apply the above Claim to choose nonempty canonical clopen sets $U_\alpha \subseteq U'_\alpha$ and $V_\alpha \subseteq V'_\alpha$ such that $\pi_I(0) = \pi_I(0)$. An easy verification shows that the sequences $\{I_\beta : \beta \leq \alpha\}$, $\{U_\beta : \beta \leq \alpha\}$ and $\{V_\beta : \beta \leq \alpha\}$ satisfy conditions (i)–(vi) for all $\beta, \gamma \leq \alpha$, thus finishing our recursive construction.

Let $K = \bigcup_{\alpha < \omega_1} I_\alpha$. By (iv), the set $G = \bigcup_{\alpha < \omega_1} U_\alpha$ is contained in $U$ and $H = \bigcup_{\alpha < \omega_1} V_\alpha$ is contained in $V$, so that $G \cap H = \emptyset$. To obtain a contradiction, it suffices to show that the sets $G$ and $H$ have a common cluster point in $\sigma(a)$. From (v), (ii) and our definition of the sets $G$ and $H$ it follows that $G = \sigma(a) \cap \pi_K^{-1}(\pi_K(G))$ and $H = \sigma(a) \cap \pi_K^{-1}(\pi_K(H))$, so we can assume without loss of generality that $K = I$.

By Lemma 2.3, the $P$-space $\sigma(a)$ is pseudo-$\omega_1$-compact. Hence the family $\gamma = \{U_\alpha : \alpha < \omega_1\}$ has an accumulation point $x \in \sigma(a)$ and every neighborhood of $x$ in $\sigma(a)$ intersects uncountably many elements of $\gamma$. Let $O$ be a canonical open neighborhood of $x$ in $X$ and let $C = \text{coord} O$. Since $|C| \leq \aleph_0$, (ii) implies that there exists $\beta < \omega_1$ such that $C \subseteq I_\beta$. There are uncountably many ordinals $\alpha < \omega_1$ such that $\beta \leq \alpha$ and $O \cap U_\alpha \neq \emptyset$. For every such an $\alpha < \omega_1$, let $z_\alpha$
be an arbitrary point of the set \( \pi_{I_\alpha}(O \cap U_\alpha) \subseteq \pi_{I_\alpha}(O) \cap \pi_{I_\alpha}(U_\alpha) \). From (iv) it follows that \( \pi_{I_\alpha}(U_\alpha) = \pi_{I_\alpha}(V_\alpha) \), so \( z_\alpha \in \pi_{I_\alpha}(O) \cap \pi_{I_\alpha}(V_\alpha) \). Choose a point \( z \in V_\alpha \) such that \( \pi_{I_\alpha}(z) = z_\alpha \). Since \( \text{coord} \mathcal{O} = C \subseteq I_\beta \subseteq I_\alpha \), we conclude that \( z \in O \cap V_\alpha \neq \emptyset \). This immediately implies that \( x \) is an accumulation point of the family \( \{V_\alpha : \alpha < \omega_1\} \) and, hence, \( x \in \overline{H} \). Thus, \( x \in \overline{G \cap H} \neq \emptyset \), which is a contradiction.

We have thus proved that \( \pi_J(U) \cap \pi_J(V) = \emptyset \) for some nonempty countable subset \( J \) of \( I \), whence it follows that \( U = \pi(a) \cap \pi_J^{-1}(U) \). In other words, \( U \) depends only on the set \( J \).

A simple modification of the argument in the proof of Lemma 3.4 (combined with the \( \Delta \)-lemma) implies the following corollary.

**Corollary 3.5.** Let \( \{X_i : i \in I\} \) be a family of \( P \)-spaces such that the product \( X = \prod_{i \in I} X_i \) is pseudo-\( \omega_1 \)-compact. If \( U \) and \( V \) are open sets in \( X \) and \( U \cap V = \emptyset \), then there exists a nonempty countable set \( J \subseteq I \) such that \( \pi_J(U) \cap \pi_J(V) = \emptyset \).

It is not clear whether one can find a countable set \( J \subseteq I \) in Corollary 3.5 satisfying \( \overline{\pi_J(U) \cap \pi_J(V)} = \emptyset \).

**Lemma 3.6.** Let \( X = \prod_{i \in I} \) be a product space and \( \sigma(a) \subseteq X \) be the \( \sigma \)-product with center at \( a \in X \). Suppose that \( \emptyset \neq J \subseteq I \) and that a continuous map \( f : X \rightarrow Y \) to a Hausdorff space \( Y \) satisfies \( f(x) = f(y) \) whenever \( x, y \in \sigma(a) \) and \( \pi_J(x) = \pi_J(y) \). Then \( f \) depends only on \( J \).

**Proof:** Let \( x, y \in X \) satisfy \( \pi_J(x) = \pi_J(y) \). Suppose to the contrary that \( f(x) \neq f(y) \) and choose in \( X \) disjoint open neighborhoods \( U \) and \( V \) of \( x \) and \( y \), respectively, such that \( f(U) \cap f(V) = \emptyset \). We can assume without loss of generality that the sets \( U \) and \( V \) are canonical and \( \text{coord} U = C = \text{coord} V \). Let us define two points \( x^*, y^* \in X \) by

\[
x^*(i) = \begin{cases} x(i) & \text{if } i \in C; \\
x^*(i) = a(i) & \text{if } i \in I \setminus C
\end{cases}
\]

and, similarly,

\[
y^*(i) = \begin{cases} y(i) & \text{if } i \in C; \\
y^*(i) = a(i) & \text{if } i \in I \setminus C.
\end{cases}
\]

Then \( x^*, y^* \in \sigma(a) \) and \( \pi_J(x^*) = \pi_J(y^*) \), so that \( f(x^*) = f(y^*) \). On the other hand, we have \( x^* \in U \) and \( y^* \in V \), whence \( f(x^*) \in f(U) \) and \( f(y^*) \in f(V) \). Since \( f(U) \cap f(V) = \emptyset \), this implies that \( f(x^*) \neq f(y^*) \), which is a contradiction.

Let \( f : X \rightarrow Y \) and \( g : X \rightarrow Z \) be continuous maps, where \( Y = f(X) \). We say that \( f \) is finer than \( g \) or, in symbols, \( f \prec g \) if there exists a continuous map \( \varphi : Y \rightarrow Z \) such that \( g = \varphi \circ f \). The theorem below is the main result of this section.
Theorem 3.7. Let $X = \prod_{i\in I} X_i$ be a product of $P$-spaces and $f: X \to Y$ be a continuous map onto a space $Y$ of countable pseudocharacter. If $X$ is pseudo-$\omega_1$-compact, then $f$ depends on at most countably many coordinates. In addition, one can find a countable set $C \subseteq I$ and, for each $i \in C$, a continuous map $h_i: X_i \to \mathbb{N}$ to the discrete space $\mathbb{N}$ such that $(\prod_{i \in C} h_i) \circ \pi_C < f$. Hence $nw(Y) \leq \aleph_0$.

Proof: First, we show that $f$ depends on countably many coordinates. Choose any point $a \in X$ and denote by $\sigma(a)$ the $\sigma$-product of the spaces $X_i$ with center at $a$. Let $\sigma(a)$ carry the relative $\aleph_0$-box topology (which is finer than the subspace topology of $\sigma(a)$ inherited from $X$). By Lemma 2.3, the $P$-space $\sigma(a)$ is pseudo-$\omega_1$-compact. Since $\psi(Y) \leq \aleph_0$, the set $F_y = f^{-1}(y) \cap \sigma(a)$ is clopen in $\sigma(a)$ for each $y \in Y$. Clearly, $\{F_y : y \in f(\sigma(a))\}$ is a partition of $\sigma(a)$ into disjoint clopen sets. Hence, the pseudo-$\omega_1$-compactness of $\sigma(a)$ implies that the image $Z = f(\sigma(a))$ is countable.

Given a nonempty set $J \subseteq I$, we denote by $\pi_J$ the projection of $X$ onto $X_J = \prod_{i \in J} X_i$. By Lemma 3.4, every set $F_y$ depends only on a countable number of coordinates, that is, there exists a countable set $C(y) \subseteq I$ such that $F_y = \sigma(a) \cap \pi_{C(y)}^{-1}(\sigma(y))(F_y)$. Put $C = \bigcup_{y \in Z} C(y)$. Then $C$ is a countable subset of $I$ and $F_y = \sigma(a) \cap \pi_{C(y)}^{-1}(\sigma(y))(F_y)$ for each $y \in Z$. Therefore, if $x, y \in \sigma(a)$ and $\pi_C(x) = \pi_C(y)$, then $f(x) = f(y)$. Apply Lemma 3.6 to conclude that $f$ depends only on the set $C$. In other words, there exists a map $f_C: X_C \to Y$ such that $f = f_C \circ \pi_C$. The map $f_C$ is continuous because the projection $\pi_C$ is open. We can assume, therefore, that $C = I$ (and $f_C = f$). In addition, we can assume that $I = \omega$, i.e., $X = \prod_{n \in \omega} X_n$ and that each factor $X_n$ is infinite.

For every $n \in \omega$, consider the subspace $K_n$ of $X$ defined by

$$K_n = \{x \in X : x(i) = a(i) \text{ for each } i > n\}.$$ 

Then $K_n \cong \prod_{i \leq n} X_i$, so that $K_n$ is a pseudo-$\omega_1$-compact $P$-space. As above, it is easy to see that the image $f(K_n)$ is countable for each $n \in \omega$ and the set $F_{n,y} = K_n \cap f^{-1}(y)$ is clopen in $K_n$ for each $y \in f(K_n)$. By Lemma 3.3, every set $F_{n,y}$ can be represented as a countable union of basic open sets of the form $U_0 \times \cdots \times U_n$, where $U_i$ is a clopen subset of $X_i$ for each $i \leq n$ (we identify $K_n$ and $X_0 \times \cdots \times X_n$). Since these representations of the sets $F_{n,y}$ involve only countably many clopen sets in each of the factors $X_0, \ldots, X_n$, one can find, for every $i \leq n$, a continuous map $g_{n,i}: X_i \to \mathbb{N}$ to the discrete space $\mathbb{N}$ such that the direct product $p_n = \prod_{i \leq n} g_{n,i}$ satisfies $p_n \prec f_n$, where $f_n = f |_{K_n}$. For every $i \in \omega$, let $g_i$ be the diagonal product of the family $\{g_{n,i} : n \geq i\}$. Then the map $g_i: X_i \to \mathbb{N}^{\omega \setminus i}$ is continuous and, clearly, the product map $q_n = \prod_{i \leq n} g_i$ satisfies $q_n \prec p_n \prec f_n$ for each $n \in \omega$. Again, the image $g_i(X_i)$ is countable and the fibers $g_i^{-1}(y)$, with $y \in g_i(X_i)$, form a partition of $X_i$ into clopen sets. Hence, for every $i \in \omega$, there exists a continuous onto map $h_i: X_i \to \mathbb{N}$ satisfying $h_i \prec g_i$. Let
$h = \prod_{i \in \omega} h_i : X \to \mathbb{N}^\omega$ be the direct product of the family $\{h_i : i \in \omega\}$. Note that each map $h_i$ is open and onto, and so is the map $h$.

Let us verify that $h \prec f$. Indeed, since $h_i \prec g_i$ for each $i \in \omega$, we have $\prod_{i \leq n} h_i \prec \prod_{i \leq n} g_i = q_n \prec f_n$ and, hence,

$$\phi_n = h_{\mid K_n} = \prod_{i \leq n} h_i \prec f_n$$

for all $n \in \omega$. First, we claim that $h^{-1}(x) \subseteq f^{-1}(f(x))$ for every $x \in X$. Suppose to the contrary that there exist points $x, y \in X$ such that $h(x) = h(y)$ but $f(x) \neq f(y)$. Choose in $Y$ disjoint neighborhoods $U_x$ and $U_y$ of $f(x)$ and $f(y)$, respectively. By the continuity of $f$, there are canonical open sets $V_x \ni x$ and $V_y \ni y$ in the product space $X$ such that $f(V_x) \subseteq U_x$ and $f(V_y) \subseteq U_y$. We can assume without loss of generality that $V_x = V_0^x \times \cdots \times V_n^x \times P_n$ and $V_y = V_0^y \times \cdots \times V_n^y \times P_n$, where $n \in \omega$, the sets $V_i^x, V_i^y$ are open in $X_i$ for $i = 0, \ldots, n$ and $P_n = \prod_{i > n} X_i$. For every $n \in \omega$, denote by $r_n$ the retraction of $X$ onto $K_n$ defined by $r_n(x)(i) = x(i)$ if $i \leq n$ and $r_n(x) = a(i)$ if $i > n$. Then $x' = r_n(x) \in V_x \cap K_n$ and $y' = r_n(y) \in V_y \cap K_n$. Therefore, from $f(x') \in f(V_x) \subseteq U_x$, $f(y') \in f(V_y) \subseteq U_y$ and $U_x \cap U_y = \emptyset$ it follows that $f(x') \neq f(y')$. By (3), however, we have $h \prec \phi_n \circ r_n \prec f_n \circ r_n = f \circ r_n$ and, hence, the equality $h(x) = h(y)$ implies that $f(r_n(x)) = f(r_n(y))$ or, equivalently, $f(x') = f(y')$. This contradiction proves the claim. So, there exists a map $i : \mathbb{N}^\omega \to Y$ satisfying $f = i \circ h$. Since the map $h$ is open, $i$ is continuous. Therefore, $h \prec f$.

Finally, the space $\mathbb{N}^\omega$ is second countable, so that the image $Y = f(X) = i(\mathbb{N}^\omega)$ has a countable network. \qed

It is shown in [15, Lemma 3.29] that every $\omega$-stable space is pseudo-$\omega_1$-compact. For $P$-spaces, $\omega$-stability and pseudo-$\omega_1$-compactness are equivalent by [15, Proposition 3.30]. It turns out that this equivalence holds for arbitrary products of $P$-spaces.

**Corollary 3.8.** Suppose that the product $X = \prod_{i \in I} X_i$ of $P$-spaces is pseudo-$\omega_1$-compact. Then the space $X$ is $\omega$-stable.

**Proof:** Let $f : X \to Y$ be a continuous map onto a space $Y$ which admits a coarser second countable Tychonoff topology. Then $Y$ is Hausdorff and $\psi(Y) \leq \aleph_0$, so that $nw(Y) \leq \aleph_0$ by Theorem 3.7. \qed

By [1, Theorem 10], every $\sigma$-product of Lindelöf $P$-spaces is $\omega$-stable. The next corollary extends this result to products of Lindelöf $P$-spaces.

**Corollary 3.9.** Every product of Lindelöf $P$-spaces is $\omega$-stable.

**Proof:** By Noble’s theorem in [10], finite products of Lindelöf $P$-spaces are Lindelöf (hence, pseudo-$\omega_1$-compact). Therefore, an arbitrary product $X = \prod_{i \in I} X_i$
of Lindelöf $P$-spaces is pseudo-$\omega_1$-compact by Lemma 2.3, and the required conclusion follows from Corollary 3.8.

In general, the product of two pseudo-$\omega_1$-compact $P$-spaces can fail to be pseudo-$\omega_1$-compact. In the class of $P$-groups, however, pseudo-$\omega_1$-compactness becomes productive by Lemmas 2.2 and 2.3. This explains, in part, the strong factorization property of products of $\mathbb{R}$-factorizable $P$-groups given in the next theorem.

**Theorem 3.10.** Let $G = \prod_{i \in I} G_i$ be a direct product of $\mathbb{R}$-factorizable $P$-groups. If $f: G \to Y$ is a continuous map onto a space $Y$ with $\psi(Y) \leq \aleph_0$, then there exists a quotient homomorphism $\pi: G \to H$ onto a second countable topological group $H$ such that $\pi \prec f$. In particular, $\text{nw}(Y) \leq \aleph_0$.

**Proof:** By Lemmas 2.2 and 2.3, the group $G$ is pseudo-$\omega_1$-compact. Apply Theorem 3.7 to find a countable set $C \subseteq I$ and, for each $i \in C$, a continuous map $h_i: G_i \to \mathbb{N}$ such that $(\prod_{i \in C} h_i) \circ \pi_C \prec f$. Since the groups $G_i$ are $\mathbb{R}$-factorizable, for each $i \in C$ there exists a continuous homomorphism $p_i: G_i \to K_i$ onto a second countable group $K_i$ such that $p_i \prec h_i$. Note that the fibers $p_i^{-1}(y)$ are $G_\delta$-sets in $G_i$, so they are open in $G_i$. Clearly, the homomorphism $p_i$ remains continuous if we endow the group $K_i$ with the discrete topology. The group $G_i$ is pseudo-$\omega_1$-compact by Theorem 1.1, so the cover of $G_i$ by the fibers $p_i^{-1}(y)$, with $y \in K_i$, is countable. Hence the discrete group $K_i = p_i(G_i)$ is countable and the homomorphism $p_i$ is open.

Let $p$ be the direct product of the homomorphisms $p_i$, $i \in C$. Then the homomorphism $p: \prod_{i \in C} G_i \to \prod_{i \in C} K_i$ is continuous, open and the group $H = \prod_{i \in C} K_i$ is second countable. It is clear that the homomorphism $\varphi = p \circ \pi_C$ of $G$ to $H$ is continuous, open and satisfies $\varphi \prec (\prod_{i \in C} h_i) \circ \pi_C \prec f$. Therefore, there exists a continuous map $i: H \to Y$ such that $f = i \circ \varphi$ and, hence, $Y = i(H)$. This implies that $Y$ has a countable network.

The following corollary to Theorem 3.10 is immediate. It was proved (by a different method) in [15].

**Corollary 3.11.** Let $G$ be a direct product of $\mathbb{R}$-factorizable $P$-groups. Then the group $G$ is $\mathbb{R}$-factorizable and $\tau$-stable for $\tau \in \{\omega, \omega_1\}$.

**Proof:** The $\mathbb{R}$-factorizability of $G$ follows directly from Theorem 3.10. In addition, $G$ is $\omega_1$-stable by [15, Theorem 3.9]. To conclude that $G$ is $\omega$-stable, apply Corollary 3.8 and Lemmas 2.2 and 2.3.

By a theorem of Comfort and Ross [5], the class of pseudocompact groups is productive. Therefore, Corollary 3.11 extends a certain similarity in the permanence properties of $\mathbb{R}$-factorizable $P$-groups and pseudocompact groups mentioned in Section 2. In addition, the groups of both classes are $\omega$-stable. In fact, one can apply Lemma 5.9 of [14] to prove the following analogue of Theorem 3.10 for
pseudocompact groups: if a regular space $Y$ of countable pseudocharacter is a continuous image of (a $G_\delta$-subset of) a pseudocompact group, then $nw(Y) \leq \aleph_0$.

4. Open problems

Here we formulate two open problems concerning Theorem 2.5.

**Problem 4.1.** Is every $\aleph_0$-bounded $P$-group topologically isomorphic to a subgroup of an $R$-factorizable $P$-group?

**Problem 4.2.** Does Theorem 2.5 remain valid in the non-abelian case?

**References**


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