Unordered Baire-like vector-valued function spaces.

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Abstract

In this paper we show that if \( I \) is an index set and \( X_i \) a normed space for each \( i \in I \), then the \( \ell_p \)-direct sum \((\oplus_{i \in I} X_i)_p\), \( 1 \leq p \leq \infty \), is UBL (unordered Baire-like) if and only if \( X_i, i \in I \), is UBL. If \( X \) is a normed UBL space and \((\Omega, \Sigma, \mu)\) is a finite measure space we also investigate the UBL property of the Lebesgue-Bochner spaces \( L_p(\mu, X) \), with \( 1 \leq p < \infty \).

In what follows \((\Omega, \Sigma, \mu)\) will be a finite measure space and \( X \) a normed space. As usual, \( L_p(\mu, X), 1 \leq p < \infty \), will denote the linear space over the field \( \mathbb{K} \) of the real or complex numbers of all \( X \)-valued \( \mu \)-measurable \( p \)-Bochner integrable (classes of) functions defined on \( \Omega \), provided with the norm

\[ ||f|| = \left\{ \int_\Omega ||f(\omega)||^p d\mu(\omega) \right\}^{1/p} \]

When \( A \in \Sigma, \chi_A \) will denote the indicator function of the set \( A \).

On the other hand, if \( \{X_i, i \in I\} \) is a family of normed spaces, we denote by \((\oplus_{i \in I} X_i)_p\), with \( 1 \leq p < \infty \), the \( \ell_p \)-direct sum of the spaces \( X_i \), that is to say:

\[ (\bigoplus_{i \in I} X_i)_p = \{ \mathbf{x} = (x_i) \in \prod\{X_j, j \in I\} : (||x_i||) \in \ell_p \} \]

provided with the norm \(||(x_i)|| = ||(||x_i||)||_p\). If \( p = \infty \), then

\[ (\bigoplus_{i \in I} X_i)_\infty = \{ \mathbf{x} = (x_i) \in l_\infty ((X_i)) : \text{card (supp } \mathbf{x} \text{) } \leq N_0 \} \]

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equipped with the norm \( \| (x_i) \| = \sup \{ \| x_n \|, n \in \mathbb{N} \} \).

A Hausdorff locally convex space \( E \) over \( \mathbb{K} \) is said to be unordered Baire-like, \([6]\) (also called UBL in \([5]\)) if given a sequence of closed absolutely convex sets of \( E \) covering \( E \), there is one of them which is a neighbourhood of the origin. When \( E \) is metrizable, \( E \) is said to be totally barrelled (also called TB in \([5]\)) if given a sequence of linear subspaces of \( E \) covering \( E \) there is one which is barrelled. This last definition coincides with the one given in \([5]\) and \([8]\) for the general locally convex case.

It is known that if \( \mu \) is atomless, \( L_p(\mu, X) \) enjoys very good strong barrelledness properties (even if \( p = \infty \)) \(([1] \text{ and } [2])\). If \( \mu \) has some atom, then \( X \) must share the same strong barrelledness property than \( L_p(\mu, X) \) do. On the other hand, by a well-known result of Lurje, \((\oplus_{i \in \mathbb{N}} X_i)_p \) is barrelled (and hence, Baire-like) if and only if each \( X_i \) is barrelled (see \([5]\), 4.9.17). This result has been extended independently in \([3]\) and \([4]\) by showing that, whenever each \( X_i \) is seminormed, \((\oplus_{i \in I} X_i)_p \) is barrelled (ultrabarrelled) if and only each \( X_i \) is barrelled (ultrabarrelled). For the definitions of Baire-like and ultrabarrelled spaces see \([5]\) (pp. 333 and 366).

In this paper we are going to investigate for a general positive \( \mu \) the UBL property of the space \( L_p(\mu, X), 1 \leq p < \infty \), whenever \( X \) is UBL. We will also prove that \((\oplus_{i \in I} X_i)_p \), with \( 1 \leq p \leq \infty \), is UBL if and only if each \( X_i \) is UBL.

**Proposition 1** If \( X \) is an UBL space, then \( L_1(\mu, X) \) is UBL.

**Proof.** Our argument is based upon the proof of the Proposition 2 of \([7]\). So, let \( \{W_n, n \in \mathbb{N}\} \) be a sequence of closed absolutely convex subsets of \( L_1(\mu, X) \) covering \( L_1(\mu, X) \). It suffices to show that there is an \( i \in \mathbb{N} \) such that \( W_i \) absorbs the family

\[
\{\chi_A x/\{\mu(A)\}, \|x\| = 1, A \in \Sigma, \mu(A) \neq 0\}
\]

since, if \( \chi_A x/\{\mu(A)\} \in qW_i \) for some \( q \in \mathbb{N} \), each \( x \in X \) of norm one and each \( A \in \Sigma \) with \( \mu(A) \neq 0 \), given any simple function \( s = \sum_{1 \leq j \leq n} y_j \chi_{C_j} \) of \( L_1(\mu, X) \), with \( C_j \in \Sigma, \mu(C_j) \neq 0, \|y_j\| \neq 0 \) for \( 1 \leq j \leq n \) and \( C_i \cap C_j = \emptyset \) if \( i \neq j \), so that \( \|s\|_1 \leq 1 \), then \( \sum_{1 \leq j \leq n} \|y_j\|/\mu(C_j) = \|s\|_1 \leq 1 \), and since \( W_i \) is absolutely convex,

\[
\sum_{1 \leq j \leq n} y_j \chi_{C_j} = \sum_{1 \leq j \leq n} \|y_j\|/\mu(C_j) \chi_{C_j}(y_j/\|y_j\|)/\{\mu(C_j)\} \in qW_i
\]

Hence, \( W_i \), being closed, absorbs the closed unit ball of \( L_1(\mu, X) \).

Let us define the closed absolutely convex subsets of \( X \)

\[
V_{nm} = \{x \in X : \chi_A x/\{\mu(A)\} \in mW_n \text{ for each } A \in \Sigma \text{ with } \mu(A) \neq 0\}
\]

for each \( n, m, n \in \mathbb{N} \).

Given \( z \in X, z \neq 0 \), then \( L(z) := \{f(z) : f \in L_1(\mu)\} \) is a closed subspace of \( L_1(\mu, X) \) isomorphic to \( L_1(\mu) \) and therefore there are \( r, s \in \mathbb{N} \) such that \( \chi_A x/\{\mu(A)\} \in sW_r \) for each \( A \in \Sigma \) with \( \mu(A) \neq 0 \). This implies that \( z \in V_{rs} \) and, consequently, that \( \bigcup\{V_{nm} : n, m \in \mathbb{N}\} = X \). As \( X \) is UBL there are \( i, j, k \in \mathbb{N} \) so that \( kV_{ij} \) contains the unit sphere of \( X \). Hence \( \chi_A x/\{\mu(A)\} \in jkW_i \) for each \( x \in X \) so that \( \|x\| = 1 \) and each \( A \in \Sigma \) with \( \mu(A) \neq 0 \). This completes the proof. \( \square \)
**Proposition 2** Let $X$ be an UBL space. If $L_p(\mu, X), 1 < p < \infty$, is a TB space, then $L_p(\mu, X)$ is UBL.

**Proof.** If $\{W_n, n \in \mathbb{N}\}$ is a sequence of closed absolutely convex subsets of $L_p(\mu, X)$ covering $L_p(\mu, X)$, a similar argument to the proof of the previous proposition shows that there exists an index $j \in \mathbb{N}$ such that $W_j$ absorbs the family

$$\{\chi_A x / \{\mu(A)\}^{1/p}, \|x\| = 1, A \in \Sigma, \mu(A) \neq 0\}.$$}

This implies that the linear span of $W_j$ contains the subspace of the simple functions. Hence span$(W_j)$ is a dense subspace of $L_p(\mu, X)$ and thus ([6], Theorem 4.1) there is no loss of generality by assuming that span$(W_n)$ is dense in $L_p(\mu, X)$ for each $n \in \mathbb{N}$.

Since we have supposed that $L_p(\mu, X)$ is TB, it follows that there exists an $i \in \mathbb{N}$ such that span$(W_i)$ is barrelled. This ensures, $W_i$ being closed in $L_p(\mu, X)$, that span$(W_i)$ is closed. Consequently, one has that span$(W_i) = L_p(\mu, X)$. This implies that $W_i$ is absorbent in $L_p(\mu, X)$. Since $W_i$ was absolutely convex and closed by hypothesis, we have that $W_i$ is a barrel in $L_p(\mu, X)$ and hence a zero-neighbourhood because $L_p(\mu, X)$ is always barrelled ([2]).

**Lemma 1** Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of normed spaces and assume that $\{W_n, n \in \mathbb{N}\}$ is a sequence of closed absolutely convex subsets of $(\oplus_{n=1}^\infty X_n)_p$ covering $(\oplus_{n=1}^\infty X_n)_p$, $1 \leq p \leq \infty$. Then there is $m \in \mathbb{N}$ such that

$$\text{span}(W_m) \supseteq (\oplus_{n=m}^\infty X_n)_p.$$}

**Proof.** If this is not the case, for each $n \in \mathbb{N}$ there is

$$x_n \in (\oplus_{k>n} X_k)_p \setminus \text{span}(W_n)$$

with $\|x_n\| = 1$. Since the sequence $(x_n)$ is bounded in $(\oplus_{n=1}^\infty X_n)_p$, then for each $\xi \in \ell_1$ the series $\sum_n \xi_n x_n$ converges to some $x(\xi)$ in the completion $(\oplus_{n=1}^\infty X_n)_p$ of $(\oplus_{n=1}^\infty X_n)_p$. Since $x(\xi)_j = \sum_n \xi_n x_{nj} = \sum_{1 \leq n \leq j-1} \xi_n x_{nj} \in X_j$, it follows that $x(\xi) / (\oplus_{n=1}^\infty X_n)_p$ and then $D = \{\sum_n \xi_n x_n, \xi \in \ell_1, \|\xi\|_1 \leq 1\}$ is a Banach disk in $(\oplus_{n=1}^\infty X_n)_p$. Consequently, there must be some $m \in \mathbb{N}$ such that $W_m$ absorbs $D$ and hence $x_m \in \text{span}(W_m)$, a contradiction. 

**Theorem 1** If $X_n$ is UBL for each $n \in \mathbb{N}$, then $(\oplus_{n=1}^\infty X_n)_p, 1 \leq p \leq \infty$, is UBL.

**Proof.** Our argument adapts some methods of [8] to our convenience.

If $(\oplus_{n=1}^\infty X_n)_p$ is not UBL, there exists a sequence $\{W_n, n \in \mathbb{N}\}$ of closed absolutely convex subsets of $(\oplus_{n=1}^\infty X_n)_p$ covering $(\oplus_{n=1}^\infty X_n)_p$ such that no $W_n$ is a neighbourhood of the origin in $(\oplus_{n=1}^\infty X_n)_p$.

Define $F = \{F \in \{\text{span}(W_n), n \in \mathbb{N}\} : \exists m \in \mathbb{N} \in F \supset (\oplus_{n=m}^\infty X_n)_p\}$. If $F$ does not cover $(\oplus_{n=1}^\infty X_n)_p$ then $(\oplus_{n=1}^\infty X_n)_p$ is covered by all those subspaces span$(W_n)$ that do not belong to $F$, as a consequence of the Theorem 4.1 of [6]. But this contradicts the previous lemma. Hence $F$ covers the whole space.
Let $\mathcal{F}_n := \{ F \in \mathcal{F} : F \text{ does not contain } X_n \}$, where we consider $X_n$ as a subspace of $(\oplus_{n=1}^{\infty} X_n)_p$. Let us see first that $\mathcal{F} = \bigcup \{ \mathcal{F}_n, n \in \mathbb{N} \}$. Indeed, if $G \in \mathcal{F}$, there is a $n(G) \in \mathbb{N}$ with $G \supseteq (\oplus_{n=n(G)} X_n)_p$. Hence, there must be $r \leq n(G)$ such that $G$ does not contain $X_r$, otherwise $G = (\oplus_{n=1}^{\infty} X_n)_p$, which is a contradiction because $G = \text{span}(W_p)$ for some $p$ and we would have that $W_p$ is a barrel, hence a zero-neighbourhood since $(\oplus_{n=1}^{\infty} X_n)_p$ is barrelled. Thus, $G \in \mathcal{F}_r$.

Let us show that considering $X_j$ as a subspace of $(\oplus_{n=1}^{\infty} X_n)_p$ there is $j \in \mathbb{N}$ such that $\cup \{ F, F \in \mathcal{F}_j \} \supseteq X_j$. Otherwise for each $j \in \mathbb{N}$ there would be some norm one $x_j \in X_j$ verifying that $x_j \notin \cup \{ F, F \in \mathcal{F}_j \}$. Defining $x_j \in (\oplus_{n=1}^{\infty} X_n)_p$ such that $x_{jk} = 0$ if $j \neq k$ while $x_{jj} = x_j$, then $(x_j)$ is a basic sequence in $(\oplus_{n=1}^{\infty} X_n)_p$ equivalent to the unit vector basis of $\ell_p$ if $p < \infty$ or $c_0$ if $p = \infty$. Hence, reasoning as in the previous lemma, we have that the closed linear span $L$ of $(x_j)$ in $(\oplus_{n=1}^{\infty} X_n)_p$, is contained in $(\oplus_{n=1}^{\infty} X_n)_p$. Since $\mathcal{F}$ covers $L \subseteq (\oplus_{n=1}^{\infty} X_n)_p$ and $L$ is a Banach space, it follows that there is some $F \in \mathcal{F}$ so that $x_j \in F$ for each $j \in \mathbb{N}$. But, as we have seen that $\mathcal{F} = \cup \{ \mathcal{F}_n, n \in \mathbb{N} \}$, there is a $k \in \mathbb{N}$ such that $F \in \mathcal{F}_k$. Therefore $x_k \in \cup \{ G : G \in \mathcal{F}_k \}$, which is a contradiction.

Finally, choose a positive integer $m$ such that $\cup \{ F : F \in \mathcal{F}_m \} \supseteq X_m$. As $X_m$ is UBL, there is $G \in \mathcal{F}_m$ with $G \supseteq X_m$. This is a contradiction, since $G \in \mathcal{F}_m$ if and only if $(G \in \mathcal{F}$ and) $G$ does not contain $X_m$. \qed

**Theorem 2** Let $I$ be a non-empty index set and let $\{ X_i, i \in I \}$ be a family of normed spaces. Then $(\oplus_{i \in I} X_i)_p$, with $1 \leq p \leq \infty$, is UBL if and only if $X_i$ is UBL for each $i \in I$.

**Proof.** If $I$ is finite, the conclusion is obvious, and if $I = \mathbb{N}$ the result has been proved in the previous theorem. Thus we may assume that card $I > \aleph_0$. If $(\oplus_{i \in I} X_i)_p$ is not UBL there exists a sequence $\{ W_n, n \in \mathbb{N} \}$ of closed absolutely convex subsets of $(\oplus_{i \in I} X_i)_p$ covering $(\oplus_{i \in I} X_i)_p$ such that no $W_n$ is a neighbourhood of the origin in $(\oplus_{i \in I} X_i)_p$. Hence there is a sequence $(x_n)$ in the unit sphere of $(\oplus_{i \in I} X_i)_p$ such that $x_n \notin \text{span}(W_n)$ for each $n \in \mathbb{N}$. As each $x_n$ is countably supported $J := \cup \{ \text{supp } x_n, n \in \mathbb{N} \}$ is a countable subset of $I$. But $(\oplus_{j \in J} X_j)_p$ is UBL as a consequence of the previous theorem, and hence there is some $m \in \mathbb{N}$ such that $W_m \cap (\oplus_{j \in J} X_j)_p$ is a neighbourhood of the origin in $(\oplus_{j \in J} X_j)_p$. Therefore $x_m \in \text{span}(W_m)$, a contradiction. \qed

**Open problem:** Assuming that $\mu$ is atomless and $X$ is a normed space, is $L_p(\mu, X)$, $1 \leq p < \infty$, a TB space?

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References


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